L	T	P	Cr
4	0	0	4

Course Code: PHY.506
Course Title: Mathematical Physics
Total Hours: 60

Learning Outcomes:
At the end of the course, students will able to:

- Learn about the transformation of coordinates, matrix algebra, complex functions, symmetry, Group theory, tensors,
- Develop a strong background to pursue research in theoretical physics.

Course Contents

Unit-I **15 hours**
Vector Calculus, Matrices & Tensors: Vector calculus: properties of Gradient, divergence and Curl, matrix algebra, Solution of linear equations. Linear transformations. Change of Basis. Caley-Hamilton theorem, Eigen values and Eigen vectors, curvilinear coordinates (spherical and cylindrical coordinates).

Unit-II **15 hours**
Tensor: Tensors, Symmetric and antisymmetric, kronecker and Levi Civita tensors.
Delta, Gamma, and Beta Functions: Dirac delta function, Properties of delta function, Gamma function, Properties of Gamma and Beta functions.

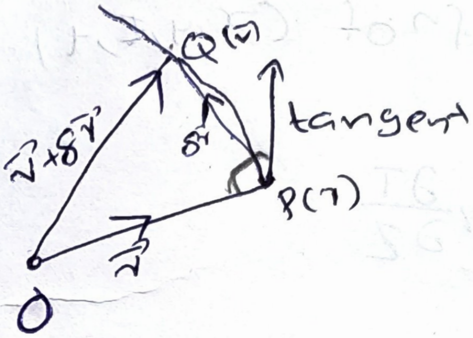
Unit-III **15 hours**
Symmetry and Elements of group theory: Symmery elements, Point groups, Character tables for some point groups and the orthogonality theorem. Group postulates, Lie group and generators, representation, Commutation relations, SU(2), O(3).

Unit-IV **15 hours**
Complex Variable: Elements of complex analysis, Analytical functions, Cauchy-Riemann equations, Cauchy theorem, Properties of analytical functions, Contours in complex plane, Integration in complex plane, Deformation of contours, Cauchy integral representation, Taylor and Laurent series, Isolated and essential singular points, Poles, Residues and evaluation of integrals, Cauchy residue theorem and applications of the residue theorem.

Mathematical physics

$$f(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$ functional form of a vector

Differntiation of Vector



time $t+\delta t$ Particle move from P to Q

$$\vec{OP} + \vec{PQ} = \vec{OQ}$$

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \delta\vec{r}$$

$$\lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} \Rightarrow = \frac{d\vec{r}}{dt}$$ $\Rightarrow$ tangent of the Path at which Particle is moving.

$r(t)$

(which is $\perp$ to $\vec{r}$)

Second order differentiation

Properies should HK class

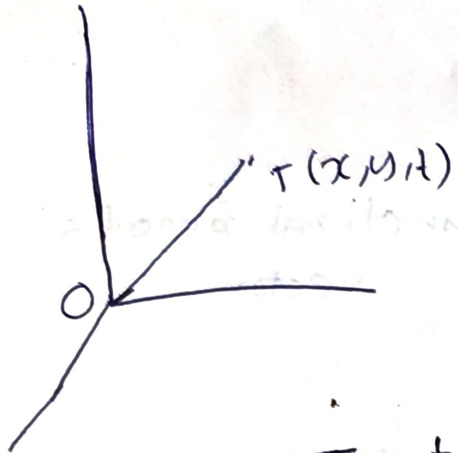
1) $\frac{d}{dt}(\vec{F}\cdot\vec{G}) = \vec{F}\cdot\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\cdot\vec{G}$

2) $\frac{d}{dt}(\vec{F}\times\vec{G}) = \vec{F}\times\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\times\vec{G}$

3) $\frac{d}{dt}(\vec{a}\,\vec{b}\,\vec{c}) = \left(\frac{d\vec{a}}{dt}\,\vec{b}\,\vec{c}\right) + \left(\vec{a}\,\frac{d\vec{b}}{dt}\,\vec{c}\right) + \left(\vec{a}\,\vec{b}\,\frac{d\vec{c}}{dt}\right)$

4) $\frac{d}{dt}\left(\vec{a}\times(\vec{b}\times\vec{c})\right) = \left[\frac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c})\right] + \left[\vec{a}\times\left(\frac{d\vec{b}}{dt}\times\vec{c}\right)\right] + \left[\vec{a}\times\left(\vec{b}\times\frac{d\vec{c}}{dt}\right)\right]$

a Candile at 'O'



at Point T temperature is a fn of (x,y,z,t)

rate of change $\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}$

mathamatically $\vec{P} = i\frac{\partial T}{\partial x} + j\frac{\partial T}{\partial y}\,j + \frac{\partial T}{\partial z}\,\hat{k}$

$= \nabla T$, gradiant

=

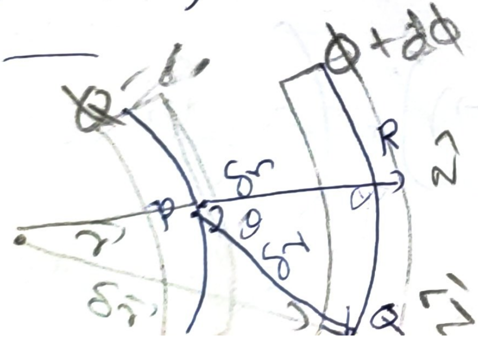
~~$\nabla\phi$ — $\nabla\phi$~~ Total change in function ϕ

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$d\phi = \nabla\vec{\phi}\cdot d\vec{r} \qquad d\vec{r} = i\,dx + j\,dy + \hat{z}\,dz$$

Equipotential Surface) level surface for potential

$\Rightarrow \phi(x,y,z) = C$, a const



$PQ = \delta r$

$\delta\vec{r} \to 0$ (ϕ and $\phi + d\phi$) on same surface $\Rightarrow d\phi = 0$

$\therefore$ $\boxed{\nabla\vec{\phi} \cdot d\vec{r} = d\phi}$ $\quad d\phi = 0$

$\nabla\vec{\phi} \cdot dr = 0 \quad \nabla\phi \perp^r dr,$ dr = tangent

so $\nabla\phi$ is $\perp^r$ to surface

$\boxed{\nabla\vec{\phi} = |\nabla\vec{\phi}|\hat{n}}$, $\hat{n}$ unit vector $\perp^r$

$\frac{d\phi}{dn}$ rate of charge of ϕ along normal

$= \lim_{\delta n \to 0} \frac{\delta\phi}{\delta n} = \lim_{\delta n \to 0} \frac{\nabla\vec{\phi} \cdot d\vec{r}}{\delta n}$

$= \lim_{\delta n \to 0} \frac{|\nabla\phi|\hat{n} \cdot d\vec{r}}{\delta n}$ $\quad$ $\hat{n} \cdot \delta\vec{r} = |\hat{n}||\delta r|\cos\theta$ $= \delta\vec{r}\cos\theta$ $= \delta n$

$= \lim_{\delta n \to} \frac{|\nabla\phi|\,\delta n}{\delta n}$ $\quad \boxed{\frac{d\phi}{d\hat{n}} = |\nabla\phi|}$ greatest change is $\perp^r$

Directional derivative $\nabla\phi$ (slope) $(|\nabla\phi|\cos\theta)$

component of ~~derivate~~ in the $\perp$ in direction of a vector $\vec{d}$ $= \vec{\nabla\phi} \cdot \hat{d}$

what direction function is changing the most.

directional derivative of ϕ in a direction of $\vec{d}$

$\frac{\partial n}{\partial r} = \cos\theta = \frac{\delta n}{\delta r}$

$\frac{\delta n}{\cos\theta} = \delta r = \frac{\delta n}{\hat{n} \cdot \hat{n}'}$

$\frac{\partial\phi}{\partial r} = \lim_{\delta r \to 0} \frac{\frac{\delta\phi}{\delta n}}{\hat{n} \cdot \hat{n}'} = \hat{n} \cdot \hat{n}' \frac{\partial\phi}{\partial n}$

$= \hat{n} \cdot \hat{n}' |\nabla\phi| = \hat{n}' \nabla\phi \quad < |\nabla\phi|$

Component of vec in a direction of another vector

$|B|\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|}$

$\frac{\vec{\nabla\phi} \cdot \vec{d}}{|\vec{d}|} = \vec{\nabla\phi} \cdot \hat{d}$

Example

~~write~~

Find the directional derivatives of $\cancel{x^2y^2z}$ $x^2y^2z^2$ at the Point $(1,1,-1)$ in the direction of the tangent to the ~~direction $x=e^t, y$~~ curve $x=e^t$, $y=\sin 2t+1$, $z=1-\cos t$ at $t=0$

$$|\nabla\phi|\cos\theta = \frac{\vec{\nabla}\phi \cdot d\vec{r}}{|\vec{r}|} = \nabla\phi \cdot \hat{r}$$

$$\nabla\phi = 2y^2z^2x\hat{i} + 2x^2yz^2\hat{j} + 2x^2y^2z\hat{k}$$

$$(1,1,-1) = 2\times1\times1\times\hat{i} + 2\times\hat{j} + 2\times-1\hat{k} = 2i+2j-2k$$

$$\vec{r} = e^t i + (\sin 2t+1)j + (1-\cos t)k$$

$$\text{tangent} = \frac{d\vec{r}}{dt} = e^t i + 2\cos 2t - \sin t k$$

$$(t=0) = i + 2j + 0$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+4} = \sqrt{5}$$

$$\frac{2i+2j+2k \cdot i+2j+0k}{\sqrt{5}}$$

$$\frac{2i+4i}{\sqrt{5}} = \frac{6}{\sqrt{5}}$$

Gradient is rate of change of ~~vector~~ of a scalar field

Gradient zero: eg:

why use gradient: $F = -\nabla V$

Divergence

$\nabla \cdot F = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial F_z}{\partial z}$, how much each component is charging.

rate of ~~of~~ change of sum of each components

Example

Continuity eqn

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

ρ - density
v - velocity

Say $\rho = \text{const.}$

$\cancel{\nabla \cdot \rho v}$ $\nabla \cdot v = 0$, $\rho = \text{const}$
liquid is incompressible
$v \Rightarrow$ Solinodial vectr fn

$$\vec{\nabla} \cdot \nabla \phi = \nabla^2 \phi \quad \text{Laplace eqn } (\nabla^2 \phi = 0)$$

Curl

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

excmpe : The linear velocity v of the particle mov with angular velocit

$$\boxed{v = \vec{\omega} \times \vec{r}}$$

Example

$x^2y^2z^2$

~~write~~ Find the directional derivatives of ~~x^2+y^2+z~~ ~~x^2y^2+v~~ at the Point $(1,1,-1)$ in the direction of the tangent to the ~~curve~~ ~~the direction $x=e^t, y$~~ curve $x=e^t$, $y=\sin 2t+1$, $z=1-\cos t$ at $t=0$

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$$\frac{2i+2j+2k\cdot i+2j+0k}{\sqrt{5}}$$

$$\frac{2+4}{\sqrt{5}} = \frac{6}{\sqrt{5}}$$

Gradient is rate of change of ~~vector~~ of a scalar field

Gradient zero: eg:

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Divergenie

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rate of ~~of~~ changed ~~of~~ sum of each components

Example

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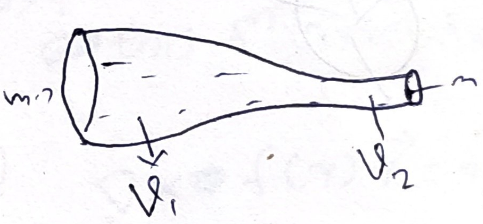


$$\vec{\nabla} \cdot \nabla \phi = \nabla^2 \phi$$ Laplace eqn ($\nabla^2 \phi = 0$)

Curl

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

excmpe: The linear velocity v of the particle mov_ with angular velocit

$$\boxed{v = \vec{\omega} \times \vec{r}}$$

$$\vec{\nabla}\times\vec{v} = \vec{\nabla}\times(\vec{\omega}\times\vec{r}) = \nabla\times\begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_2 \\ x & y & z \end{vmatrix}$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z-\omega_3 y) & (\omega_1 z-\omega_3 x) & (\omega_3 y-\omega_2 z) \end{vmatrix}$$

$$\vec{\nabla}\times\vec{v} = \underline{2\vec{\omega}}$$

Curl operator result in ~~rotation~~ rotational vectr.

$\nabla\times V = 0 \Rightarrow$ irrotational

Curl of central force Field

$f(r)\hat{r} \Rightarrow$ central force

$$\hat{r} = \frac{x}{|x|} i + \frac{y}{|y|} j + \frac{z}{|z|} z$$

$$\vec{\nabla}\times\vec{F}(r)\hat{r}\Big|_x = \frac{\partial}{\partial y}\frac{z}{r}F(r) - \frac{\partial}{\partial z}\frac{y}{r}f(r) \qquad \Big|\ |\vec{r}| = r$$

x component

$$= z\left(\frac{d}{dr}\frac{F(r)}{r}\right)\left(\frac{\partial r}{\partial y}\right) - y\left(\frac{d}{dr}\frac{F(r)}{r}\right)\left(\frac{\partial r}{\partial z}\right)$$

$$= z\left(\frac{d}{dr}\frac{F(r)}{r}\right)\cdot\left(\frac{y}{r}\right) - y\left(\frac{\partial}{\partial r}\frac{F(r)}{r}\right)\left(\frac{z}{r}\right)$$

$$= \frac{zy}{r}\left(\frac{d}{dr}\frac{f(r)}{r}\right) - \frac{zy}{r}\frac{d}{dr}\frac{f(r)}{r}$$

Home work

① Find the normal to the surface $x^2+y^2=z$ at a point $(1,2,5)$

2) If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, show that ① $\vec{\nabla}|\vec{r}| = \frac{\vec{r}}{|\vec{r}|}$ and $\vec{\nabla}\left|\frac{1}{\vec{r}}\right| = -\frac{\vec{r}}{|\vec{r}|^3}$

③ For electric potential $\phi = \frac{e}{r}$ find $\nabla^2\phi = ?$

4) Find the value of n for which the vector $r^n\vec{r}$ is solenoidal. Given, $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

5) Show that $\vec{\nabla}\cdot(\phi\vec{A}) = \vec{\nabla}\phi\cdot\vec{A}+\phi(\vec{\nabla}\cdot\vec{A})$

6) A vector field is given by $\vec{F} = (x^2-y^2+x)\hat{i}-(2xy+y)\hat{j}$. is this f^n irrotational? If so find it's ~~vector~~ vector potential

7) $\nabla\times f(r)\vec{r} = ?$

8) $\vec{\nabla}r^n = ?$

curl of central filed

$$\vec{\nabla}\times f(r)\hat{r} = \Big|$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x}{|\vec{r}|}i + \frac{y}{|\vec{r}|}j + \frac{z}{|\vec{r}|}k$$

$$\vec{\nabla}\times f(\vec{r})\hat{r}\Big|_x = \frac{\partial}{\partial y}\left(\frac{z}{r}f(r)\right) - \frac{\partial}{\partial z}\left(\frac{y}{r}f(\vec{r})\right)$$

x compont.

$$= z\frac{d}{dr}\frac{f(r)}{r}\frac{\partial r}{\partial y} - y\frac{d}{dr}\frac{f(r)}{r}\frac{\partial r}{\partial z}$$

cheak this again

$$\vec{r} = xi + yj + z\hat{k}$$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2+z^2}} = \frac{y}{r}$$

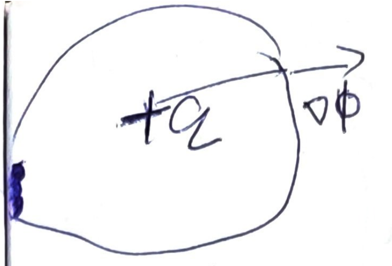
$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= z\frac{d}{dr}\frac{f(r)}{r}\frac{y}{r} - y\frac{d}{dr}\frac{f(r)}{r}\frac{z}{r}$$

$$= \frac{zy}{r}\frac{d}{dr}\left(\frac{f(r)}{r}\right) - \frac{zy}{r}\frac{d}{dr}\frac{f(r)}{r} = 0$$

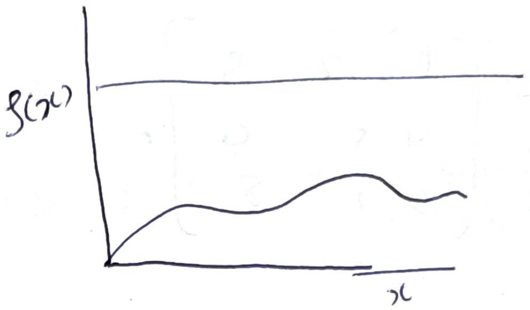
also other components $= 0$

curl of central force field $= 0$



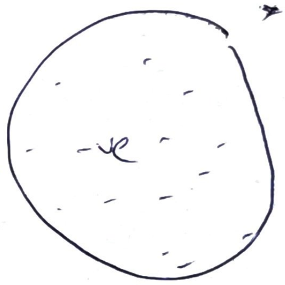
$$\frac{\partial \phi(r)}{\partial \theta}, \frac{\partial \phi(r)}{\partial \phi}, \cancel{\neq 0}\ \frac{d\phi}{dr} = 0$$

example for $\nabla\phi = 0$



$\frac{\partial f}{\partial x} = 0$

$\frac{\partial f}{\partial x} \neq 0$



* Pool of electron (Homogenias electron gas)

$\rho = \text{Const.}$

here. $\cancel{\nabla\phi} = \nabla\rho = 0$

$\Rightarrow H_2$ molecule. $\nabla\rho \neq 0$ any other real system

Curl of velocity

$$\vec{\nabla} \times \vec{v}\Big|_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial v_y}$$

$\vec{v} = v_x i + v_y j$

$\frac{\partial v_y}{\partial x} \Rightarrow +ve$

$\Rightarrow 2\omega$

$\frac{\partial v_x}{\partial y} = -ve$

Matrices

$$\left.\begin{array}{l} a_1 x_1 + a_2 x_2 = h_1 \\ b_1 x_1 + b_2 x_2 = h_2 \end{array}\right\} \Rightarrow \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$m \times n$ = row x Column

$m = n$ = Squre matrix

$$\begin{bmatrix} A & B & C \\ D & E & F \\ H & I & J \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix} =$$

matrix Algebra

$AB \neq BA$ Non, Commutative

$(AB)' = B'A'$

$(x, p) \neq 0$ c

$$\overset{A}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \overset{B}{\begin{bmatrix} 4 & 5 & 6 \end{bmatrix}} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

4 5 6

8

$$\overset{B}{\begin{bmatrix} 4 & 5 & 6 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = 32$$

Transpose of a matrix

compex

$$A = [a_{ij}] \Rightarrow A^T = [a_{ji}] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

if $A = A'$ (symmetric matrix)

eg: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{bmatrix}$

Adjoint of a matrix

$A(n \times n)$ matrix $\Rightarrow$ Cofactor $(-1)^{i+j} M_{ij}$

$C = M_{ij}$ are minors matrix co

$\text{Adj} A = C^T$ Transpose of cofactor matrix

Inverse of a matrix $= A^{-1} = \frac{\text{Adj} A}{|A|}$

If $|A| = 0$ then A^{-1} not defind so it is called Singular matrix

Trace Sum of the diagonal elements

Row / colomn matrix (only row or column)

null matrix $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Square matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Diagonal matrix $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Scalar matrix $= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \end{bmatrix}$

symmetric $(A^T = A)$

$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

skew symmetric $(A^T = -A)$

$\begin{bmatrix} 0 & -h & g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$

set of linear eqn

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = h_1$

$a_{21}x_1 + a_{22}x_2 + \dots \quad a_{2n}x_n = h_2$

$- \quad - \quad - \quad - \quad - \quad -$

$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = h_n$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}}_{h}$$

$Ax = h$

$x = A^{-1}h$

Triangular matrix

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}}_{\text{UperT}} \quad \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}}_{\text{LT}}$$

Orthogonal matrix $(A^T A = I)$

Conjugate of mat

$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}, \bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$

$A^{\Theta} = (\bar{A})'$? $\bar{A} = (A^{\Theta})'$ wrong

Unitary matri $(A^{\Theta}A = I)$

Hermitian $(A = \bar{A})$

skew " $(A = -\bar{A})$

idempotent $(A^2 = A)$

Periodic $(A^{k+1} = A)$

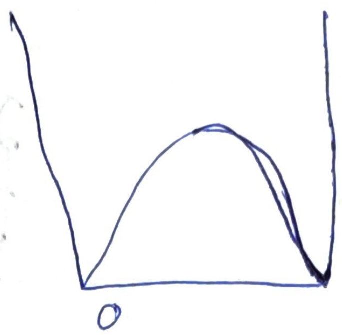
Nillpotent $(A^k = 0)$

Involuntary $(A^2 = I)$

Singular $(|A| = 0)$

(no inverse)

Particle in a box



$n=1$

$$\psi = \sqrt{2/\ell}\, \sin \frac{n\pi x}{\ell}$$

matrix element for P & p^2 for particle in a box?

Adjoint

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_4 & d_5 \end{bmatrix} \quad (4\times4)$$

$$C = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \\ D_1 & D_2 & D_3 & D_4 \end{bmatrix}$$

$$\text{Adj}\, A = C^T$$

$$A_1 = \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix}$$

$$A_2 = -\begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix}$$

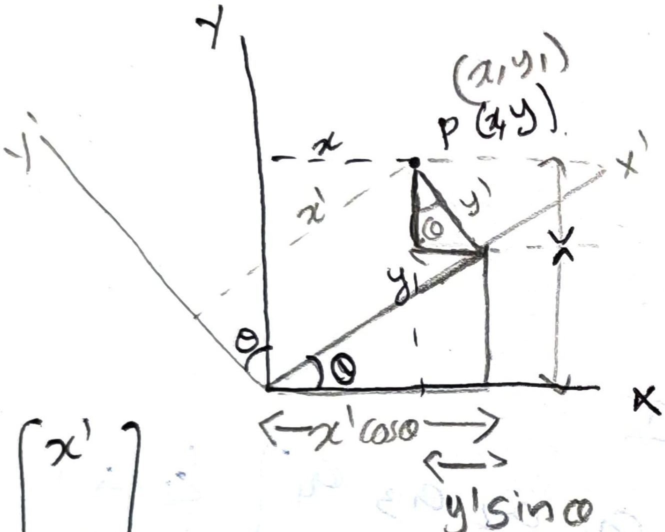
~~Diagonalisation of matrix~~ ?

$\cancel{D = P^{-1} A P}$

$\cancel{D \neq D^{-1} \quad D^2 \neq I}$

Inverse of D = $\frac{[D]}{\text{trace}}$?

Rotation of coordinate System,



$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

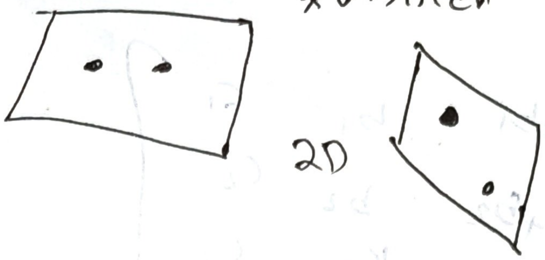
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

rotation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj} A}{|A|} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

*Unitcell



2D

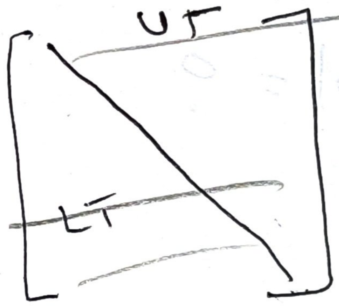
for staking use rotation matrix

Gauss elimination methode [Calculation of Determinant]

$A = (a_{ij})$

$$D_n = \sum_{\substack{j=1 \\ i=1}}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

make UT=0 or LT=0



$\Rightarrow$ elementary tranformation (ET)

($PA = B$, $A \sim B$ equi valent)

1) Can interchang 2 rows ~~and~~ or 2 columns

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} \Rightarrow \underset{B}{\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}}$$

$A \simeq B$

2) multiply all the elements of a row or column by non-zero Scalar

$K \cdot R_{ij}$

3) You can add two rows or column

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & b_1 & c_1 \\ a_2+b_2 & b_2 & c_2 \\ a_3+b_3 & b_3 & c_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + \begin{bmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{bmatrix}$$

$$|A| = 0 \quad \& \quad |B| = 0$$

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1) Find Inverse

2) Find Rank

3) Find determinant

~~Elementary matrix~~

~~$I + ET = Em$~~

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + 3R_1} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = EM$$

Inverse of A matrix (Gauss-jordan) method

eg: some elementary tranformation = P

$PA = I \Rightarrow P = A^{-1}$

$AI = A$

$AIP = AP$

$PA = I$

eg: $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, $A^{-1} = ?$

$A = IA$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

~~$R_1 \to \frac{R_1}{3}$~~

$R_1 \to R_1/3$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

~~$R_2 \to R_2 - 2R_1$~~

$R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 4/3 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 4/3 \\ 0 & +1 & -4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ +2/3 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_2 \to \frac{R_2}{-1}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 4/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A \qquad R_3 \to R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 4/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ -2 & +3 & -3 \end{bmatrix} A \qquad R_3 \to R_3 \times -3$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \qquad R_1 \to R_1 - \frac{4}{3}R_3, \quad R_2 \to R_2 + \frac{4}{3}R_3$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \qquad R_1 \to R_1 + R_2$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$$

Rank

At least one non-zero minor with order γ

no. of linearly independent rows

$$A = \begin{bmatrix} I_\gamma & 0 \\ 0 & 0 \end{bmatrix} = \text{rank } \gamma$$

I

$\left[\overset{II}{(I_\gamma, 0)}, \overset{III}{\begin{bmatrix} I_\gamma \\ 0 \end{bmatrix}}, \overset{IV}{I_\gamma} \right]$ normal form or Canonical form

rank of a matrix is the order of the largest square matrix whose determinant is not zero

Rank of a matrix by Triangular form

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} = \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank 2 = [no. of non zero rows in UT matrix]

(R only ?)

column-space - is the span of columns of your matrix (basis vector)

Rank : no. of Dimentions in Column space.

Rank of the matrix by Normal form
(Canonical form)

R & C allowed

eg: $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & 11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ $R_3 \to R_3 - R_2$, $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 + C_1$, $C_4 \to C_4 - 3C_1$ $\Rightarrow$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_4 \to R_4 + \frac{1}{2}R_3$, $C_3 \to C_3 + \frac{6}{7}C_2$, $C_4 \to C_4 - \frac{11}{7}C_2$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $C_4 \to C_4 + 2C_3$, $R_2 \to -\frac{1}{7}R_2$, $R_3 \to -\frac{1}{2}R_3$

Rank of A = 3 //

Gauss elimination methode to find Determinance.

row or Column transformation

$$D = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

$\det(cA) = c^n |A|$

$\det(A^T) = \det A$

$$= 3 \begin{vmatrix} 1 & 2/3 & \frac{1}{3} \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 3\times 2 \begin{vmatrix} 1 & 2/3 & 1/3 \\ 1 & 3/2 & 1/2 \\ 1 & 1 & 4 \end{vmatrix}$$

Aim

To make lower triange = 0

$$D = \begin{bmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{bmatrix} = \quad |D| = \lambda_1 \lambda_2 \lambda_3$$

$(R_2 \to R_2 - R_1);(R_3 \to R_3 - R_1)$

$$3\times 2 \begin{vmatrix} 1 & 2/3 & 1/3 \\ 0 & 5/6 & 1/6 \\ 0 & 1/3 & 11/3 \end{vmatrix}$$

$$\frac{6}{11} \begin{vmatrix} 1 & \frac{2}{3} & 1/3 \\ 0 & 54/6 & 1/6 \\ 0 & 11/3 & 11/3 \end{vmatrix}$$

$C_2 - C_2 - C_3$

$$\frac{6}{11}\begin{vmatrix} 1 & 2\frac{1}{3} & 2/3 \\ 0 & 54/6 & 1/3 \\ 0 & 0 & 11/3 \end{vmatrix}$$

$$|D| = 1 \times \frac{6}{11} \times \frac{54}{6} \times \frac{11}{3} = 18$$

every row swithch mutipy by (-1)

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix} \Rightarrow (-1)\begin{vmatrix} 1 & 1 & 4 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix} \Rightarrow (-1)\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & -7 \\ 0 & -1 & -11 \end{vmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\Rightarrow (-1)\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & -7 \\ 0 & 0 & -18 \end{vmatrix}$$

$R_3 \to R_3 + R_2$

First step next session

$$D = (-1) \times (-18) = 18$$

Solution of linear equations

$$a_1 x_1 + a_{12} x_2 + \cdots a_{n} x_n = b_1$$

$$-\quad-$$

$$a_{1n} x_n + a_{2n} x_2 + \cdots a_{mn} x_n - b_n$$

$$A x = B, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Create a augmented matrix.

$$C = [A : B] \quad \begin{bmatrix} a_{11} & a_{12} & \cdots a_{1n} & : b_1 \\ - & - & \cdots & : \\ a_{1n} & a_{2n} & \cdots a_{nn} & : b_n \end{bmatrix}$$

1) Rank of A = Rank of C $\Rightarrow$ Unique Solution

2) rank A is less than Rank of C — ∞ Solution (n−r) arbitary consts.

=

eg: $x + 2y = 4$ $\quad x = -1$ $\quad$ Rank = 2

$3x + 2y = 2$ $\quad y = 5/2$

$x + 2y = 4$

$3x + 6y = 2$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \Rightarrow \text{Rank } 1 \quad \text{Not consistent}$$

infinite

$x + 2y = 4$

$3x + 4y = \cancel{8}5$

Rank of $A = 1$

ank of $C = 2$

$A \neq C \Rightarrow$ are in consistant.

Solution of Set of linear equation.

Gauss - Jordon methode

* Only Row transformation only.

$x - y + 2z = 3$

$x + 2y + 3z = 5$

$3x - 4y - 5z = -13$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix}$$

$$\begin{pmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$\lambda_3 z = b_3$

$z = b_3 / \lambda_3$

$R_2 - R_2 - R_1$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -13 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & -1 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -22 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -32 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -64 \end{bmatrix}$$

$-32z = -64 \Rightarrow z = 2$

$3y + z = 2$

$3y + 2 = 2 \quad \Rightarrow y = 0$

$x - y + 2z = 3$

$x - 0 + 4 = 3$

$x = -1$

Crammers rule

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Solve

$$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX=B \qquad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$X=A^{-1}B$$

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow xD=\begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

$\cancel{xD=a_1x \quad b_1}$

$$C_1 \to C_1 + C_2 \times y + C_3 z$$

$$\begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & d_3 & c_3 \end{vmatrix} = D_1$$

$$x = \frac{D_1}{D}$$

$$y = \frac{D_2}{D}$$

$$z = \frac{D_3}{D}$$

$$\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = D_2$$

$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = D_3$$

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eg: $\left.\begin{aligned} 5x - 7y + z &= 11 \\ 6x - 8y - z &= 15 \\ 3x + 2y - 6z &= 7 \end{aligned}\right\}$ Solve using cramer and G.J methode

also find det $A = \begin{bmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{bmatrix}$

Hw

Using G. elimination method.

Ans: Gauss-Jordan methode

$$\begin{bmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 19 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -7/5 & \frac{1}{5} \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11/5 \\ 15 \\ 7 \end{bmatrix} \quad R_1 \to R_1/5$$

$$\begin{bmatrix} 1 & -7/5 & 1/5 \\ 0 & 2/5 & -11/5 \\ 0 & 31/5 & -33/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11/5 \\ 9/5 \\ 21/5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 6R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -7/5 & 1/5 \\ 0 & 1 & -11/2 \\ 0 & 31/5 & -33/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11/5 \\ 9/2 \\ 2/5 \end{bmatrix} \quad R_2 \to R_2 \times 5/2$$

$$\begin{bmatrix} 1 & -7/5 & 1/5 \\ 0 & 1 & -11/2 \\ 0 & 0 & \frac{55}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11/5 \\ 9/2 \\ \frac{-55}{2} \end{bmatrix} \quad R_3 \to R_3 - 31/5\, R_2$$

$$\cancel{z}\ \frac{1675}{\cancel{67}}\ \cancel{z} = \frac{-1375}{\cancel{5}} = z = \frac{-1375}{\cancel{1675}} = \frac{-55}{67}$$

$$\begin{bmatrix} 1 & -7/5 & 1/5 \\ 0 & 0 & -11/2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11/5 \\ 9/2 \\ \end{bmatrix}$$

$= \dfrac{-4}{264}$

$z = -1$

$y = -1$

$x = 1$

55
55
605

2680
670
9380

6600
6600
72600

670
1005
1005
111220

55610
55610
611710

Using Crammers rule

$$A = \begin{bmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{bmatrix} \quad B \begin{bmatrix} 11 \\ 15 \\ 7 \end{bmatrix}$$

$$D_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix} = 55$$

$$D_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix} = \cancel{-220}\ -55$$

$$D_3 \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix} = -55$$

$$D = |A| = \ = \cancel{-415}\ 55$$

$$x_1 = \frac{D_1}{D} = \frac{\cancel{55}}{\cancel{-415}} = \cancel{\frac{-11}{83}}\ 1$$

$$y = \frac{D_2}{D} = \frac{\cancel{220}}{415} = \cancel{\frac{44}{83}}\ -1$$

$$z = D3/D = \frac{+55}{-415} = \frac{11}{83}\ -1$$

Det of A using Gauss elimination method.

$$\cancel{1 \times 1 \times 2 \times \frac{55}{10}} = \cancel{55/2}$$

$$\begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -5 \end{vmatrix} \Rightarrow 5 \begin{vmatrix} 1 & -7/5 & 1/5 \\ 6 & -8 & -1 \\ 3 & 2 & -5 \end{vmatrix}$$

$$\Rightarrow 5 \begin{vmatrix} 1 & -7/5 & 1/5 \\ 0 & 2/5 & -11/5 \\ 0 & 31/5 & -33/5 \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 - 6R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\Rightarrow 2/5 \times 5 \begin{vmatrix} 1 & -7/5 & 1/5 \\ 0 & 1 & -11/2 \\ 0 & 31/5 & -33/5 \end{vmatrix} \Rightarrow 2 \times \begin{vmatrix} 1 & -7/5 & 1/5 \\ 0 & 1 & -11/2 \\ 0 & 0 & 55/2 \end{vmatrix}$$

$$\therefore \Delta = \frac{55}{2} \times 2 = 55$$

Eigen vectors and Eigenvalue

$A =$ operator

$\psi =$ vector (wave function)

$$\boxed{A\psi = \lambda\psi}, \quad \lambda = \text{scalar}$$

→ Eigenvector

→ Eigenvalue

$$A\psi - \lambda\psi = 0$$

$$\boxed{(A - \lambda I)\psi = 0}$$

$$|A - \lambda I| = 0$$

Scalar determinant

$$H\psi = E\psi$$

$$H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad \lambda = \lambda_1 \lambda_2 \lambda_3$$

$$|H - \lambda I| = 0 \Rightarrow \left| \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$-\lambda(-2\lambda + \lambda^2) - 1(2 - \lambda) = 0$$

$$\Rightarrow 2\lambda^2 - \lambda^3 - 2 + \lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

$$\lambda^2(2 - \lambda) - 1(2 - \lambda) = 0$$

$$(2 - \lambda)(\lambda^2 - 1) = 0$$

$$\lambda = \pm 1, 2,$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{bmatrix} \begin{pmatrix} -c \\ 2c \\ -c \end{pmatrix} = b \begin{pmatrix} c \\ -2c \\ c \end{pmatrix}$$

A

$Ax = \lambda x$

$(A - \lambda I)x = 0$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c \\ -c \\ 0 \end{bmatrix}}_{x} = b \underbrace{\begin{bmatrix} -c \\ c \\ 0 \end{bmatrix}}_{x}$$

Find Eigen value of A for such eqn

$c_1 \quad c_2 \quad c_3$

$x_1 \quad x_2 \quad x_3$

$Ax = -bx$

Eigen value $-\underline{b}$

Eigen vector $= \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

~~$c_1 + c_2$~~ $= 0$

$c_3 = 0$

$(A - \lambda I)c = 0$

$$\begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\underline{\lambda = 2}$ //

$-2c_1 + c_2 = 0$

$c_1 - 2c_2 = 0$

$(2-2)c_3 = 0 \Rightarrow 0 = 0$

$\Rightarrow$

$c_1 = c_2 = 0$

$c_3 = c$

Eigen vector. $\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$ with eigen value 2

$\lambda = +1$ $\quad -c_1 + c_2 = 0 \qquad c_1 = c_2 = c$

$c_1 - c_2 = 0$

$c_3 = 0$ eigen vector $\begin{pmatrix} c \\ c \\ 0 \end{pmatrix}$

$\lambda = -1 \quad c_1 + c_2 = 0$

$c_1 + c_2 = 0$ Eigen vector $= \begin{pmatrix} c \\ -c \\ 0 \end{pmatrix}$

$c_3 = 0$

- Norm of the vector or $\begin{pmatrix} (c & c & 0) \end{pmatrix} \begin{pmatrix} c \\ c \\ 0 \end{pmatrix} =$

$A A^T$

given vector $= (c \quad c \quad 0) \Rightarrow$

convert to magnitude. $\Rightarrow \sqrt{c^2 + c^2 + 0^2}$ / also called norm of the vector.

$= c\sqrt{2}$

Norm = 1

$\hookrightarrow$ normalized vector (I)

How to convert it into normalized vector

$\frac{1}{c\sqrt{2}} (c \quad c \quad 0) = \frac{1}{\sqrt{2}} (1 \quad 1 \quad 0) = \text{I}$

norm

whether Eigen vectors orthogonal or not.

orthogonal or Not,

$\Rightarrow \vec{A}, \vec{B}, (c \ c \ 0)$, $\vec{A} = \hat{i}c + \hat{j}c + 0$

$\vec{A}\cdot\vec{B} = 0$ orthogonal.

$A \cdot B \cos 90 = 0$

$\vec{A} = \hat{i}c + \hat{j}c + 0\hat{k}$

$\vec{B} = \hat{i}c - \hat{j}c + 0\hat{k}$ $\quad \vec{A} \perp \vec{B}$

$\vec{A}\cdot\vec{B} = c^2 - c^2 = 0$

$A = (c \quad c \quad 0)$ $\quad \vec{A}\cdot\vec{B} = (c \ c \ 0)\begin{pmatrix} c \\ -c \\ 0 \end{pmatrix} = 0$

$B = (c \quad -c \quad 0)$

$(0 \quad 0 \quad c)\begin{pmatrix} c \\ c \\ 0 \end{pmatrix} = 0$ $\quad \Big| \quad (0 \ 0 \ c)\begin{pmatrix} c \\ -c \\ 0 \end{pmatrix} = 0$

orthogonal matrix (column vectors are $\boxed{\text{orthonormal}}$)

$\boxed{A A^T = I}$ $\Rightarrow \boxed{S^{-1} = S^T}$

$(c \ c \ 0)\cdot\begin{pmatrix} c \\ c \\ 0 \end{pmatrix} = 2c^2$ $\quad \frac{1}{\sqrt{2}}(1 \ 1 \ 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\times 2 = 1$

Normalized

Only holds for square matrix.

Because;
$$\begin{pmatrix} - & \ominus & \vdots \\ - & - & - \\ \cdot & \cdot & - \end{pmatrix} \begin{pmatrix} - & - & - \\ \ominus & - & - \\ - & - & - \end{pmatrix} = I$$

$$\left.\begin{aligned} C_1 &= (0 \quad 0 \quad c) \\ C_2 &= (c \quad c \quad 0) \\ C_3 &= (c \quad -c \quad 0) \end{aligned}\right\}$$ Eigen vectors ~~Norma~~ (orthoganal)

$$\left.\begin{aligned} C_1 &= \frac{1}{\sqrt{2}}(0 \quad 0 \quad 1) \\ C_2 &= \frac{1}{\sqrt{2}}(1 \quad 1 \quad 0) \\ C_3 &= \frac{1}{\sqrt{2}}(1 \quad -1 \quad 0) \end{aligned}\right\}$$ Normalized form (orthonormal)

$$S = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}, \quad S^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

$$\boxed{SS^{-1} = I}$$

$AA^T = I \Rightarrow A^T = A^{-1}$ because column vectors are orthonormal

on diagonal element $AA^T = \delta_{ij} \begin{cases} i=j, 1 \\ j \neq i, 0 \end{cases}$

$i=j$ so 1

else 0 $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Properties of Eigen Values

1) $\alpha \begin{pmatrix} c \\ c \\ 0 \end{pmatrix} \Rightarrow \lambda\alpha$

$\Downarrow$

$AX = \lambda X$ vector

$X \rightarrow \alpha X$

$A(\alpha X) = (\lambda\alpha)X$

Need check

$A \rightarrow \alpha A = A'$

$AX = \lambda X$

$(\alpha A)X = \alpha\lambda X$

$A'X = (\alpha\lambda)X$

$A'X = \lambda' X$

$\lambda' = \alpha\lambda$

If we multiply a scalar quatity with operator Eigen value also multiplied with same scalar.

2) $AX = \lambda X$

$A^m X = \lambda^m X$

3) A & A^T

$A \rightarrow \lambda$, $A^T = \lambda$

Diag

Trace of A = Trace of A^T

$\lambda_1 + \lambda_1 + \lambda_3$

$\lambda = \lambda_1, \lambda_2, \lambda_2$

$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 \; \lambda_2 & \lambda_3 \end{pmatrix}$, $A^T = ?$

Diagonalization = ?

$|A-\lambda I| = 0$

$|A^T-\lambda I| = 0$ λ is the Eigen Value.

Cayley-Hamilton theorm

$A \to (n \times n)$ matrix, eigen values are $\lambda_1, \lambda_2, \ldots \lambda_n$

$|A-\lambda I| = 0$

$$\Rightarrow A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ - & - & - & - \\ a_{n1} & - & - & a_{nn} \end{pmatrix}$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots \\ - & - & - \\ a_{n1} & & a_{nn}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1)^n \lambda^n + k_1 \lambda^{n-1} + \cdots + k_n = 0$$

(Characteristic eqn)

Cayley-Hamilton theorm

Every squre matrix satisfies its own charcteristic equation.

Proof)

$(-1)^n A^n + k_1 A^{n-1} + \cdots k_n = 0$ (BS areuded)

multiply by A^{-1}

$(-1)^n A^{n-1} + k_1 A^{n-1} + \cdots k_n A^{-1} = 0$

$$\Rightarrow A^{-1} = -\frac{1}{k_n}\left[(-1)^n A^{n-1} + A_2 A^{n-2} + \cdots\right]$$

Example $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

$$\begin{vmatrix} (1-\lambda) & 4 \\ 2 & (3-\lambda) \end{vmatrix} = (3 - \lambda - 3\lambda + \lambda^2) - 8$$

$=$

$A^2 - 4A - 5 = 0 \qquad \times A^{-1}$

$A - 4I - 5A^{-1} = 0$

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 5A^{-1}$$

$$\frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} = A^{-1}$$

Hermitian Matrix (conjugate + transpose)

$H = (H^*)^T$, transpose of complex conjugate

$$H = \begin{pmatrix} 1 & 2-i \\ 2+i & 0 \end{pmatrix}, \quad H^* \begin{pmatrix} 1 & 2+i \\ 2-i & 0 \end{pmatrix}$$

$$(H^*)^T = \begin{pmatrix} 1 & 2-i \\ 2+i & 0 \end{pmatrix} = H$$

$H^\dagger$, dagger ~~H~~

In Harmitian matrix diagonal elemets ~~sho~~ must not be Complex.

$$H = \begin{pmatrix} i & 2-i \\ 2+i & 0 \end{pmatrix}$$

$$(H^*)^T = \begin{pmatrix} -i & 2-i \\ 2+i & 0 \end{pmatrix} \neq H$$

$$H^* = \begin{pmatrix} -i & 2+i \\ 2-i & 0 \end{pmatrix}$$

Eigen value of ~~of~~ Hermitian matrix.

Diagonalizatin =

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \quad \lambda \end{bmatrix}$$

So diagonal elemet canot be imeginary.

Because ~~Diagonal ele~~. Eigen values canot be Compex.

A, v_1 λ_1 - Eigen val -

v_2 λ_2

~~AA^T=~~ $Av_1 = \lambda_1 v_1$ | $Av_2 = \lambda_2 v_2$

~~multiply both side by~~

$(Av_1)^{\dagger} = (\lambda_1 v_1)^{\dagger}$

mutiply with v_2

$(Av_1)^{\dagger} v_2 = (\lambda_1 v_1)^{\dagger} v_2$

$(A^*)^T = v_1^* (A^*)^T \cdot v_2 = \lambda_1^* v_1^* v_2$

$\Rightarrow v_1^* A v_2 = \lambda_1^* v_1^* v_2$

$= v_1^* \lambda_2 v_2 = \lambda_1 v_1^* \cdot v_2$

$\Rightarrow (\lambda_2 - \lambda_1^*) \cdot v_1^{\dagger} v_2 = 0$

| $(AB)^T = B^T A^T$
| $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$

if $2 = 1$

$\underline{(\lambda_1 - \lambda_1^*)} (v_1^{\dagger} v_p) = 0$

$\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \cdot (\cdot \quad \cdot \quad \cdot)$ = Norm of the vector.

$\Downarrow$ $\parallel$

$\parallel 0$

0

If this is zero there is no vector because it give it's length

$\Leftarrow$ $H = {}^{\dagger}H$

$(\lambda_2 - \lambda_1)(v_1^{+} v_2 = 0$ because $\lambda_1 = \lambda^*$?

$\lambda_1 \neq \lambda_2 \rightarrow$ distinct

$v_1^T v_2 = 0 \mid \lambda_2 - \lambda_1 \neq 0$

v_1 & v_2 are orthogonal. $\lambda_1, \lambda_2 =$ real

For Hermitian matrix

$H\psi = E\psi$ | Hamiltonian is actually Hermition

Basis set

~~Hermition~~

eg: $\begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix} \Rightarrow$ whether this is hermitian or not.

$(A^*)^T = A$ then Hermitian

Complex Conjugate.

$H^{+} = H \Rightarrow$ then Hermitian

Not Hermition.

eigen Values of Hermition

it should be real. (Because Diagonal elements real. If diagonal elements are Not real it should be Hermitian)

Properties of Eigen vectors ?

Eigen Vectors are orthogonal.

Diagonalization

Basis function

$\hat{x}, \hat{y}, \hat{z}$ ($\vec{x}, \vec{y}, \vec{z}$)

$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

$\hat{i}, \hat{j}, \hat{k}$, are Called Basic Vectors.

i, j, k, must be indipendent so orthogonal

$\vec{A} = A'_x \hat{x} + A'_y \hat{y} + A'_z \hat{z}$ $(\hat{x}, \hat{y}, \hat{z})$

Any unknown Vector Can be represented in a known Bassis vector or function,

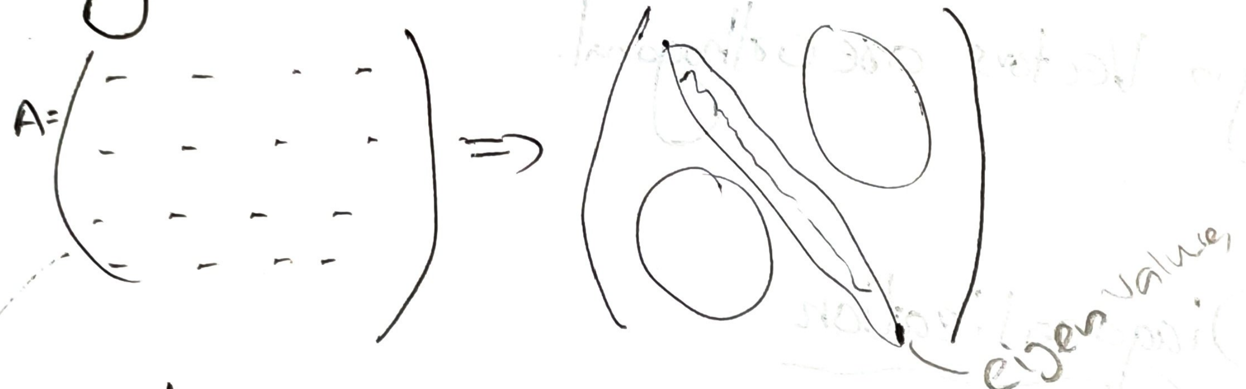
The Coe depends on the unknown Vector.

$\vec{A} = A_x \hat{x}_1 + A_y \hat{x}_2 + A_z \hat{x}_3$

$\vec{B} = B_x \hat{x}_1 + B_y \hat{x}_2 + B_z \hat{x}_3$

It is Not Nessocary that Basis should be unit Vectors.

Diagonalization



we needmatrix P to work on A to trasformed to Diagonal form

→ Eigen Vector are linearly indipendod (normalized)

Squre matrid.

Similarity
$\hat{A}=$ $P^{-1}AP$ transformtz $P=$ eigen vectors (normalized)

tarnsformation can give yo Diagonal matrix.

let A be $n\times n$ matrix, $\lambda_1, \lambda_2 \cdots \lambda_n$

k $n=3$, $\lambda_1, \lambda_2, \lambda_3$

corresponding x_1, x_2, x_3
eigen vectr

$Ax_1=\lambda_1 x_1$, $Ax_2=\lambda_2 x_2$, $Ax_3=\lambda_3 x_3$

$P=[x_1, x_2, x_3]$

~~$[Ax_1\ Ax$~~

~~$AP=[Ax_1 x_2$~~ A

$AP = A[x_1, x_2, x_3]$

$AP=[Ax_1\ Ax_2\ Ax_3]$

$= [\lambda_1 x_1, \lambda_2 x_2\ \lambda_3 x_3]$

x_1

$$X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \quad X_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

$$AP = (\lambda_1 X_1 \quad \lambda_2 X_2 \quad \lambda_3 X_3)$$

$$= \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}}_{P} * \underbrace{\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}}_{D}$$

$AP = PD \quad \times P^{-1}$

$\lambda_1 \lambda_2 \lambda_3$ are eisen vals

$APP^{-1} = PP^{-1} D$

$D = P^{-1} A P$

$A \Rightarrow \lambda_1 \lambda_2 \lambda_3 \quad , 1, 2, 3$

$A \Rightarrow$ diagonalized $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$ λ are $(2, 5, 6)$

eg:
Any electronic system

$H\psi = E\psi$

$$H = \begin{vmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{vmatrix}$$

↳ Energy states of the system

E_1, E_2, E_3

---

1) Find P for $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to be diagonalize

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$, Eigen values $= -2, 3, 6$

Eigen vector $=$ normalized form $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$(A - \lambda \underline{I})x = 0$

$$(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$$

$$P = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Show P can diagonalize A

$$P^{-1} = \frac{1}{6}\begin{pmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} = ?$$

---

linearly depended.

eg: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{pmatrix}$ can you digonalize this

$|A| = 0$

HW

Eigen vector = $AX = \lambda X$

$(A - \lambda I)X = 0$

$D = P^{-1}AP$

$D^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P$

$D^n = P^{-1}A^nP \qquad \times P$

$PD^n = PP^{-1}A^nP$

$PD^nP^{-1} = A^n PP^{-1} = I$

$$\boxed{A^n = PD^nP^{-1}}$$

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 7 \\ 3 & 1 & 1 \end{pmatrix} \Rightarrow D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$x_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$A^4 = ?$

$$P = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$D^4 = \begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 6^4 \end{bmatrix}$$

$$= \begin{pmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{pmatrix}$$

$$A^4 = PD^4P^{-1}$$

$$A^4 = \begin{pmatrix} 251 & 484 & 235 \\ 485 & 1051 & 485 \\ 235 & 485 & 251 \end{pmatrix}$$

Normalization

$norm_a = \sqrt{(a,a)} = \|a\|$

$\sqrt{a^T a} = \sqrt{a_1^2 + a_2^2 + \dots}$

Normalized Vector, → norm = 1

$(a,b) = a \cdot b = a^T b$

$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$ If $\neq 1 \Rightarrow$ A is not normalized.

If norm = 1 It is unit vector, → normalized

To Normalize,

$$\frac{\vec{A}}{|\vec{A}|} = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \hat{i} + \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \hat{j} + \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \hat{k}$$

→ norm of this vector, $\hat{n} = \frac{\vec{A}}{|\vec{A}|}$

$= n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$

$= \sqrt{n_x^2 + n_y^2 + n_z^2} = 1$

$X = ai + 3j - 5k$ find value of a

$Y = 2i - 5j + ak$ X and Y

$|\vec{X}| = \sqrt{a^2 + 34} = 1$ — (1)

$|\vec{y}| = \sqrt{a^2 + 29} = 1$ — (2)

① $\Rightarrow a^2+34=1$

$a^2=-33$

$a=\pm\sqrt{33}\,i$

② $\Rightarrow \pm\sqrt{28}\,i$

Curvelinear Co-ordinates

$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = b\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

$\vec{a}\times\vec{b}\times\vec{c} = \vec{b}(\vec{a}\cdot\vec{b}) - \vec{c}(\vec{a}\cdot\vec{b})$

$r(t) \Rightarrow \overset{\text{tangent}}{\dfrac{dr}{dt}}$ $\perp^r$ b dr / $\nabla\phi$ is $\perp^r$ to ϕ

$d\phi = \nabla\phi\cdot dr$

Directional derivative. $\Rightarrow$ along $\vec{d}$

$\nabla\phi\cdot\hat{d} \Rightarrow \dfrac{\vec{\nabla}\phi\cdot\vec{d}}{|\vec{d}|} = \nabla\phi\cos\theta.$

orthogonal matrix det( ) = 1

Eigenvale — Symmetric — real
skew — i or 0
orthogonal — complex $|\ | = 1$

Hermiti ~~unitary~~, skew, unitary

orthogonal vector ~~vector~~

Similarity transformation — orthonormal $\longrightarrow$ unitary

$D = P^{-1}AP$, same eigen values

$A^n = PD^nP^{-1}$

unitary system

$a_j\cdot a_k = a_j^{\dagger}a_k = \begin{cases} 0 & j\neq k \\ 1 & j=k \end{cases}$ ~ orthonormal

$\hat{T}_{u_j} = \dfrac{\partial\vec{r}}{\partial u_j}$

$h_j = \left|\dfrac{\partial\vec{r}}{\partial u}\right.$

$d\vec{r} = \sum_j$

$= \sum_j h$

$dV = (h_1 du$

$= h$

$\dfrac{\partial\vec{r}}{\partial u_1}$.

J of x, y, z

u

$d\vec{A}_{23} =$

$ds^2 = d\vec{r}$

$\vec{\nabla} = \sum_j \dfrac{1}{h_j}$

~~Chamakor~~ Singh

1) Find the normal to the surface $x^2+y^2=z$ at a point $(1,2,5)$

$\phi = x^2+y^2-z$

$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-z)$

$= 2x\hat{i}+2y\hat{j}-\hat{k}$

2) If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, show that

1) $\nabla|\vec{r}| = \frac{\vec{r}}{|\vec{r}|}$, 2) $\vec{\nabla}\left|\frac{1}{\vec{r}}\right| = \frac{-\vec{r}}{r^3}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$, $\nabla = \hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$

$\vec{\nabla}|\vec{r}| = \frac{\hat{i}}{2\sqrt{x^2+y^2+z^2}}\times 2x + \frac{\hat{j}\times y}{\sqrt{x^2+y^2+z^2}} + \frac{z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{|\vec{r}|}$

b) $\left|\frac{1}{\vec{r}}\right| = \frac{|1|}{|\vec{r}|}$, $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\frac{1}{|\vec{r}|} = (x^2+y^2+z^2)^{-1/2}$

$\vec{\nabla}\left(\frac{1}{|\vec{r}|}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\times 2x\hat{i} - y(x^2+y^2+z^2)^{-3/2}\hat{j}$
$- z(x^2+y^2+z^2)^{-3/2}\hat{k}$

$= \frac{-(x\hat{i}+y\hat{j}+z\hat{k})}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{-\vec{r}}{|\vec{r}|^3}$

3) for electrostatic Potential $\phi = \frac{e}{r}$ find $\nabla^2 \phi$

$r = \sqrt{x^2+y^2+z^2}$ $\quad \nabla^2 = \left(\hat{i}\frac{\partial^2}{\partial x^2} + \hat{j}\frac{\partial^2}{\partial y^2} + \hat{k}\frac{\partial^2}{\partial z^2}\right)$

$$\phi = e(x^2+y^2+z^2)^{-1/2}$$

$$\nabla\phi \frac{\partial \phi}{\partial x} = -\frac{e}{2}(x^2+y^2+z^2)^{-3/2} \times 2x\,\hat{i} - e(x^2+y^2+z^2)^{-3/2} \times y\,\hat{j}$$

$$- e(x^2+y^2+z^2)^{-3/2} z\,\hat{k}$$

$$= \frac{-e(x\hat{i}+y\hat{j}+z\hat{k})}{\left[\sqrt{(x^2+y^2+z^2)}\right]^3} = \frac{-e\vec{r}}{|\vec{r}|^3}$$

$$\nabla\phi = \frac{-e(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{-3/2}}$$

$$\boxed{\nabla^2\phi = \nabla\cdot\nabla\phi}$$

$$\nabla^2\phi = -e\left[\frac{\partial}{\partial x}\left(x/(x^2+y^2+z^2)^{3/2}\right) + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial z}\left(\frac{z}{(\sqrt{x^2+y^2+z^2})^3}\right)\right]$$

$$= e\left[\frac{(x^2+y^2+z^2)^{-3/2} \times 1 - x \times \frac{3}{2}(x^2+y^2+z^2)^{-5/2} \times 2x}{(x^2+y^2+z^2)^{-6/2}}\right.$$

$$+\left(\frac{1}{(|\vec{r}|)^3}\frac{3y^2}{|\vec{r}|^5}\right)\times|\vec{r}|^6+\left(\frac{1}{|\vec{r}|^3}-\frac{3z^2}{|\vec{r}|^5}\right)(|\vec{r}$$

$$= e\left[\cancel{\frac{|\vec{r}|^5+3x^2|\vec{r}|^3}{|\vec{r}|^5}}\right]$$

$$= e\left\{\left[\frac{|\vec{r}|^6}{|\vec{r}|^3}-\frac{3x^2|\vec{r}|^6}{|\vec{r}|^5}\right]+\left\{|\vec{r}|^3-3y^2|\vec{r}|\right\}+\left\{|\vec{r}|^3-3\right.\right.$$

$$|\vec{r}|^2$$

$$= e\left\{3|\vec{r}|^3-3|\vec{r}|\left[x^2+y^2+z^2\right]\right\}$$

$$= e\left\{3|\vec{r}|^3-3|\vec{r}|^3\right\} = 3e(r^3-r^3)$$

$$= 0$$

$$= \cancel{3e|\vec{r}|^2\left\{1+|\vec{r}|\right\}}$$

$$= 3\,\cancel{e\,(x^2+y^2+z^2)\left(1+\sqrt{x^2+y^2+z^2}\right)}$$

Find out the value of n for which the vector $r^n \vec{r}$ is solinoidal Given $\vec{r} = xi + yj + zk$

$\nabla \cdot \phi = 0 \Rightarrow \phi$ is solinoidal

$\nabla \cdot (r^n \vec{r}) = ?$ $\quad \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$r^n = (x^2+y^2+z^2)^{n/2}$ & $\vec{r} = (xi + yj + zk)$

$\nabla \cdot (r^n \vec{r}) = \nabla \cdot (r^n x i + r^n y j + r^n z k)$

$= (x^2+y^2+z^2)^{n/2} + x \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \times 2x$

$+ (x^2+y^2+z^2)^{n/2} + y^2 n (x^2+y^2+z^2)^{\left\{\frac{n-2}{2}\right\}}$

$+ (x^2+y^2+z^2)^{n/2} + nz^2 (x^2+y^2+z^2)^{\left[\frac{n-2}{2}\right]} = 0$

$= 3(x^2+y^2+z^2)^{n/2} + (x^2+y^2+z^2)^{\left[\frac{n-2}{2}\right]} n \left[x^2+y^2+z^2\right]$

$= 3(x^2+y^2+z^2)^{n/2} + n(x^2+y^2+z^2)^{n/2}$

$= (x^2+y^2+z^2)^{n/2}(3+n) = 0$

$n+3 = 0 \quad n = -3$

5) show that $\vec{\nabla}\cdot(\phi\vec{A}) = \vec{\nabla}\phi\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A})$ / $\phi(\nabla\cdot\vec{A}) + (\nabla\phi)\cdot$

$= \cancel{\phi\nabla\cdot\vec{A} + \vec{A}\cdot\nabla\phi}$

$$\vec{\nabla}\cdot(\phi\vec{A}) = \vec{\nabla}\cdot(\phi A_x i + \phi A_y j + \phi A_z \hat{k})$$

$$= \frac{\partial \phi A_x}{\partial x} + \frac{\partial \phi A_y}{\partial y} + \frac{\partial \phi A_z}{\partial z}$$

$$= \phi\frac{\partial A_x}{\partial x} + A_x\frac{\partial \phi}{\partial x} + \phi\frac{\partial A_y}{\partial y} + A_y\frac{\partial \phi}{\partial y} + \phi\frac{\partial A_z}{\partial z} + A_z\frac{\partial \phi}{\partial z}$$

$$= (\nabla\vec{\phi})\cdot A + (\nabla\cdot\vec{A})\phi$$

---

6) A vector field is given by $(x^2-y^2+x)i - (2xy+y)j$ is this field irrotational If so find its vector potential

If irrotational $\nabla\times\vec{F} = 0$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ (x^2-y^2+x) & -(2xy+y) & 0 \end{vmatrix}$$

$$= \hat{i}\left(0 - \frac{\partial}{\partial z}(2xy+y)\right] - \hat{j}\left(0 - \frac{\partial}{\partial z}(x^2+y^2+x^2)\right) + \hat{k}\left(\frac{\partial}{\partial x}\overline{(2xy+y)} - \frac{\partial}{\partial y}(x^2-y^2+x)\right]$$

$$= \hat{k}(-2y + 2y) = 0$$

$F = \nabla\phi$

$$\phi = \int (x^2 - y^2 + x)\,dx - (2xy+y)\,dy$$

$f \quad \frac{x}{3} - y^2x + \frac{x^2}{2} - xy^2 - \frac{y^2}{2} + C$

$\vec{\nabla} \times f(r)\vec{r} =$ $\quad f(r)(\vec{\nabla} \times \vec{r}) + \nabla f(r) \times \vec{r}$

$\cancel{= \nabla f(r) \times \vec{r} + \vec{r}}$

$= f(r)\underbrace{(\nabla \times \vec{r})}_{=0} + [\nabla f(r)] \times \vec{r}$

$$\vec{\nabla} \times \vec{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \cancel{i(0)}\ j\ 0$$

$$\nabla f(r) \times \vec{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} f(r) & \frac{\partial}{\partial y} f(r) & \frac{\partial}{\partial z} f(x) \\ x & y & z \end{vmatrix}$$

$$\frac{\partial}{\partial x} f(r) = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x$$

$$\Rightarrow \frac{\partial}{\partial x} = \frac{x}{|\vec{r}|}$$

$$\frac{\partial f(r)}{\partial y} = \frac{y}{|\vec{r}|}, \quad \frac{\partial}{\partial z}\ \frac{z}{|\vec{r}|}$$

$\cancel{\nabla f(r) \times \vec{r}}$

$$(\nabla f(r)) \times \vec{r} = \begin{vmatrix} i & j & k \\ \frac{x}{|\vec{r}|} & \frac{y}{|\vec{r}|} & \frac{z}{|\vec{r}|} \\ x & y & z \end{vmatrix}$$

$$= i\left(\frac{yz}{|\vec{r}|} - \frac{zy}{|\vec{r}|}\right) - j(0) + 0 = 0$$

$$= 0$$

8) $\vec{\nabla} \times r^n$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2+z^2)^{n/2} & (x^2+y^2+z^2)^{n/2} & (x^2+y^2+z^2)^{n/2} \end{vmatrix}$$

$$\cancel{= i\left[\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \times 2y\right) - j\left(\frac{n}{2}|\vec{r}|^{n/2-1} \times 2\right.}$$
$$\left.\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \times z2\right]$$

$$= \hat{i}\left(\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \times 2y - \frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \times z2\right)$$

$$- \hat{j}\left(nx|\vec{r}|^{n/2-1} - nz|\vec{r}|^{n/2-1}\right) + \hat{k}\left(nz|\vec{r}|^{n/2-1} - ny|\vec{r}|^{n/2-1}\right)$$

$\cancel{+\hat{k}(nz|\vec{r}|^{n/2-2}}$

$= ny$

Properties of Determinants

1) $|A| = |A^T|$

2) If two row or columns interchaned ~~value~~ sign of the determinats $\Delta_1 = -\Delta_1$

3) if two rows or colums are equal $|A| = 0$

4) If any row or colum mutiply by scalar, determinat also mutiply by scalor

5) The value of determinat remain unchaned if to the elements of one row (or column) be added const. mutiple of the corresponding elements of another row (or column) respectivey.

Liniear Combination

$2a + 3b =$ If we fix one the end will trace a line.

Linearly dependended

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\times$ scalar $\Rightarrow$ linearly depended

Scalars are numbers which ~~are~~ scales the vector

Linear Tranformation

Lines remain lines
Orgin remain fixed
Grid lines remain paralell and evenly spaced

Propertieties
$L(v+w) = L(v) + L(w)$
$L(cw) = c\,L(w)$

a vector ~~3e~~ $3\hat{i} + 4\hat{j}$, $\hat{i}$ & $\hat{j}$ are basis

$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ matrix form.

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} = 3\hat{i} + 4\hat{j}$$

when we apply linear transformation
$\hat{i}$ end endup in $\begin{bmatrix} a \\ b \end{bmatrix}$ then tranformation
$\hat{j}$ end up in $\begin{bmatrix} c \\ d \end{bmatrix}$ can be represented by,

$$\begin{array}{cc} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \\ \hat{i} \quad \hat{j} \end{array}$$

, then If we want to find where vector $\begin{bmatrix} x \\ y \end{bmatrix}$ after transformation will land up ~~in~~

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ b \end{bmatrix} + y \begin{bmatrix} c \\ d \end{bmatrix}$$

$$= \cancel{xa} \; \cancel{by} \begin{bmatrix} xa + yc \\ xb + yd \end{bmatrix}$$

~~mult~~

1

If trasformation $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ happend first

then $\begin{bmatrix} e & g \\ f & h \end{bmatrix}$ 2 then where vector $\begin{bmatrix} x \\ y \end{bmatrix}$

will end up

$$\begin{bmatrix} e & g \\ f & h \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

operation order

~~is called vect~~

mutiplicat

$$\begin{bmatrix} e & g \\ f & h \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow \begin{bmatrix} e & g \\ f & h \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} e & g \\ f & h \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}$$

i j

1 2

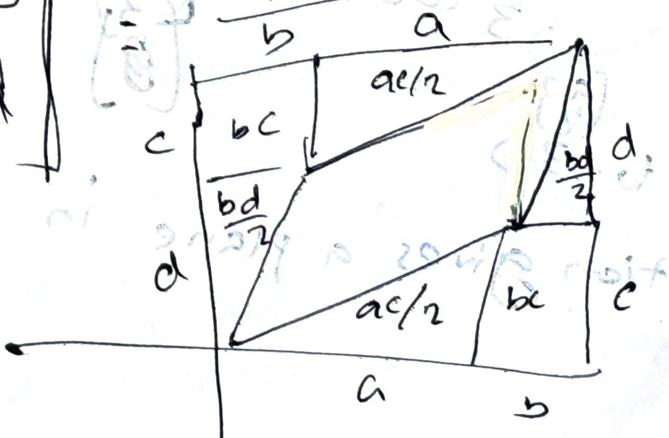
$$\begin{bmatrix} e \\ f \end{bmatrix} a + \begin{bmatrix} g \\ h \end{bmatrix} b \quad \begin{bmatrix} e \\ f \end{bmatrix} c + \begin{bmatrix} g \\ h \end{bmatrix} d \Rightarrow$$

$$\begin{bmatrix} ae + bg & ec + gd \\ af + bh & fc + hd \end{bmatrix}$$

Determinand

is the area ~~of the~~ or volume of basis vector
How it scales an arbitrary area
If D is -ve area is flipped.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$



If $|A| = \text{Zero}$, then it is ~~ap~~ tranformed into plane or a ~~---~~ line

Rank

Column space is span of ~~colum~~ columns (Basis vector) of your matrix.

Rank: No. of dimention of column space.

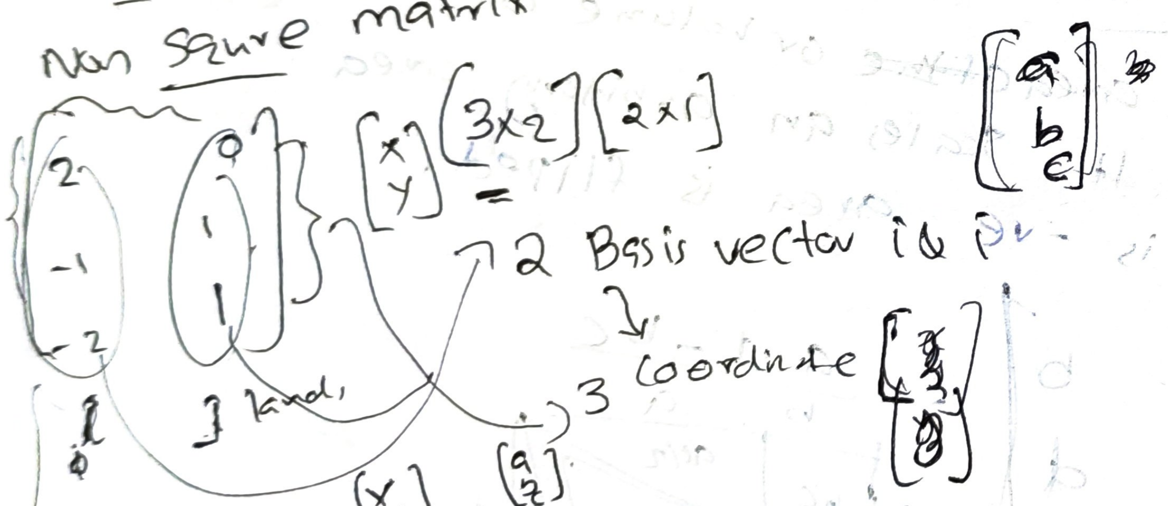
Full rank = Rank = Column space.

Null space, or kernel => set of vectors which ~~rank~~ land on origin after tranformation (~~ker~~

A => linear tranformation

A^{-1} => inverse of that $\boxed{I}$

Non square matrix



This transformation gives a plane in 3D space.

Dot product

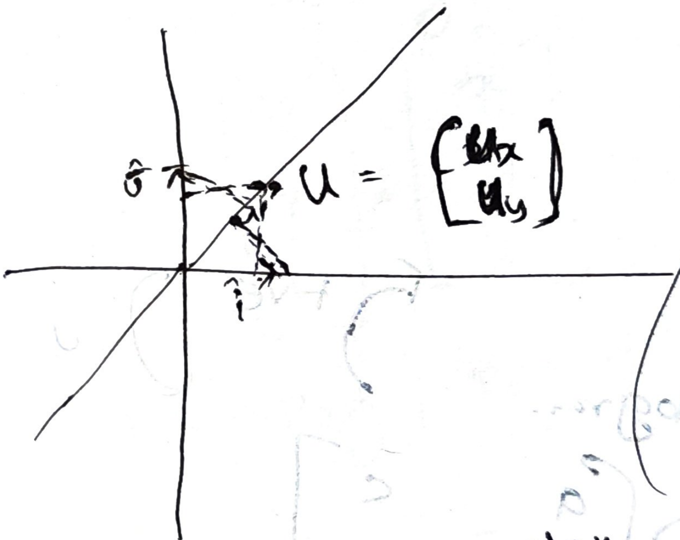
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} x_1 & y_1 \end{bmatrix}}_{\text{Transformation}} \underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\text{vector}}$$

Linear transformation from mutiple Dimention to 1D (number)

Linear tranformation depends on wher $\hat{i}$ & $\hat{j}$ lands in this case. It will be a single number.

$$\underbrace{\begin{bmatrix} \underset{\hat{i}}{a} & \underset{\hat{j}}{b} \end{bmatrix}}_{\text{Transformation}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{vector}} = [a]x + [b]y = ax + by$$

$$\begin{bmatrix} \underset{\hat{i}}{a} & \underset{\hat{j}}{b} \end{bmatrix}$$



$u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$

where do $\hat{i}$ and $\hat{j}$ land.

$\begin{bmatrix} u_x & u_y \end{bmatrix}$

because projection of $\hat{i}$ on u is equivalant to projection of u on $\hat{i}$ which is u_x

For arbitary ~~matrix~~ vector in space $\begin{bmatrix} x \\ y \end{bmatrix}$

The projection transformation $\begin{bmatrix} u_x & u_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$u_x x + u_y y$

For non unit vector

$\begin{bmatrix} 3u_x \\ 3u_y \end{bmatrix}$ matrix (transformation) associated with the vector $\Rightarrow \begin{bmatrix} 3u_x & 3u_y \end{bmatrix}$

and $\hat{j}$ to 3 times where they landed before.

- Projecting any vector on the [redacted]
multiplying with 3

Any time we have one of these linear transformation whose output space is numberline
There gonna be a unique vector $\vec{V}$ corresponding to that transformation. Appling the tranformation is same as taking dot product with that vector

$$\underbrace{\begin{bmatrix} a & b \end{bmatrix}}_{\substack{\text{vector associated}\\\text{with transformation}}}^{\text{transform to}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{2D vector}} = \text{output in line 1D} \iff \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

duality

Cross product

+ve -ve

$A \times B$ = area of the paralalogram.

$v \times w$ = determinant $\Rightarrow \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$v = \begin{bmatrix} a \\ b \end{bmatrix}, w = \begin{bmatrix} c \\ d \end{bmatrix}$

$ad - bc =$

Cross Product is not a number $\Rightarrow$ so area is the modulus and direction is given by $v \times w$

1. Define a 3d to 1d linear transformation in terms of $\vec{v}$ & $\vec{w}$
2. find dual vector
3. show that dual is $\vec{v} \times \vec{w}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is variable

function is linear

$$[?\ ?\ ?]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{array}{ccc} & \overbrace{}^{\vec{v}} & \overbrace{}^{\vec{w}} \\ x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{array}\right)$$

1x3 matrix encoding the 3d-to-1d linear transformation

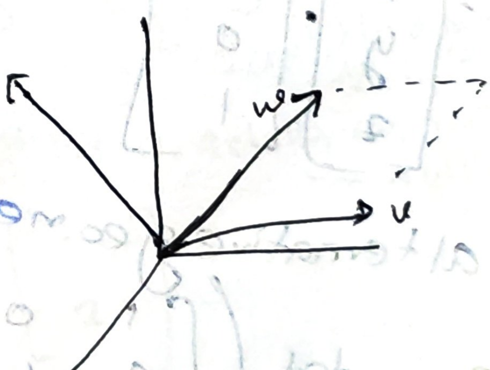
by duality

$$\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{\vec{p}} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$p_1 x + p_2 y + p_3 z = x(v_2 \cdot w_3 - v_3 \cdot w_2) + y(v_3 \cdot v_1 - v_1 \cdot w_3) + z(v_1 \cdot v_2 - v_2 w_1)$$

Geometrically

$\vec{p} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, Project ~~vector~~ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ on $\vec{p}$ & multiply the length of projection by length of $\vec{p}$



Volcume ~~of~~ of the ~~v~~ parallelpiped.

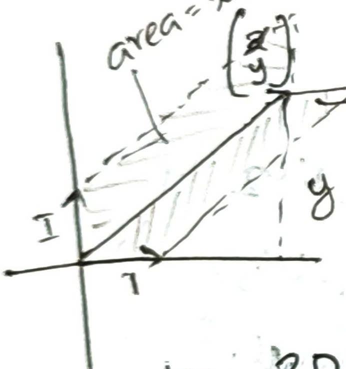
~~Find the area of $\vec{v} \times w$~~

(Area of paralallogram $(v \times w)$) $\times$ Component of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ perpendicular to v & w)

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot$ vector perpendicular to v ~~x~~ and w

Crammers rule

If $T(\vec{V}) \cdot T(\vec{W}) = \vec{V} \cdot \vec{W}$, orthonomal transformation



sighed,
area $= 1 \times y$
$= y$ = cordinate

In 3D

$$z = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Projection of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ on ~~8~~ 3rd std basis vector

(dot product)

alternative geometric interpretat.. sighed ~~area~~ volume of the paralelpipred formed by ~~basis~~ 1st and $\hat{i}, \hat{j}$ ~~bs~~ basis vector

$$z = \det\left(\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{bmatrix}\right)$$

$y = \det\begin{pmatrix} 1 & x & 0 \\ 0 & y & 0 \\ 0 & z & 1 \end{pmatrix}$, $x = \det\begin{pmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{pmatrix}$

If we apply $\underbrace{\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}}_{A}$ Transformation, all of the areas will be changed by det(A)

Signed area = det(A) y (after transformation)

$y = \frac{\text{Area}}{\det(A)}$ $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

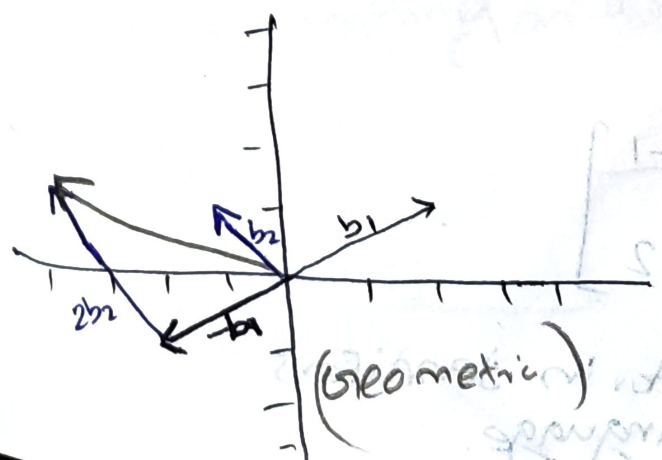
$= \frac{\det\begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix}}{\det\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}}$

Basic Vectors

$\hat{i}, \& \hat{j} \rightarrow b_1, b_2$ $\longrightarrow$ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow -1b_1 \; 2b_2$ → our co-ordinate

Intransformed co-
(numerical method) $-b_1 + 2b_2 \Rightarrow -1 \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{b_1} + 2 \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b_2} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ → In our coordinates

This is actally matrix mutiplicat



(Geometric)

$\underbrace{\begin{bmatrix} \underbrace{2 \\ 1}_{b_1} & \underbrace{-1 \\ 1}_{b_2} \end{bmatrix}}_{\text{Transformation matrix}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \rightarrow$ In TC

After transformation same linear combination of new basis vector

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$$

inver change of basis matrix | written in our cordinate | transformed co-ordinate

our to → Transformed

90°

$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\hat{i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\hat{j} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

The transformation matrix = $\begin{array}{cc} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \hat{i} \quad \hat{j} \end{array}$

our choice of basis vectors

Recording their landing spots in our own coordinate system

~~How would jennifer~~

How To translate matrix. → where her basis vects in are in our cordinate

$$\begin{array}{cc} b_1 & b_2 \end{array}$$
$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

change of basis matrix. | vector in jennifers language.

How To translate a matrix

$$\underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}}_{\text{Inverse change of Basis matrix}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Tranformation matrix in our language}} \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}}_{\text{Change of basis matrix}} \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\text{vector in Jennifer's language}} = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Same vector but in our language (Tranformed)

b_1 b_2

Her basis vector in our language.

Transformation matrix

Transformed vector but in our languge.

Transformed vector in her languge.

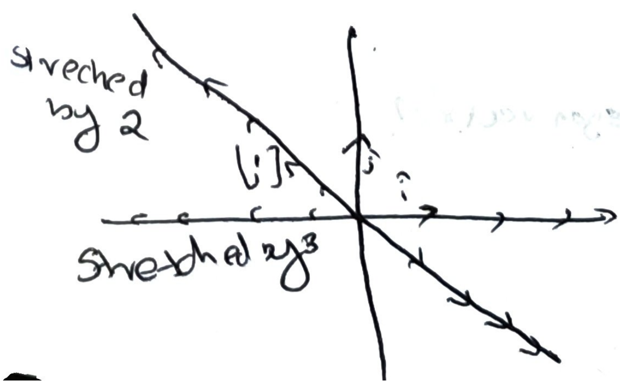
An expression like this represent empathy.

~~$A^{-1}MA$~~ $A^{-1}MA$ → Transformation of some kind as you see it

Shift in perspective.

Eigen value

Vectors staying on the span after tranformation



$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

any other vector get rotatedd.

$\boxed{\text{eigen vectors stay on the span}}$ and Scaled by eigen vale

if eigen value (-ve) => fiped.

is eigen vale $(0<\lambda<1)$ => ~~squ~~ squishod

eg: for 3D ratatation Eigen value are axis of rotation (For ratation $\lambda = 1$) (becave rotin never stretch or squish)

$$\underset{\text{Tranformat}}{A}\ \underset{\text{Eigen vect}}{\vec{v}} = \underset{\text{Eigenvale}}{\lambda}\vec{v}$$

$$\lambda \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$\overbrace{\boxed{A - \lambda I}}^{\text{transformat.}} \underbrace{\vec{v}}_{\text{vector}} = 0$$

Only way $(A-\lambda I)\vec{v}$ is zero is thate ~~det(A - λI)~~ = 0

after transformation It squishes space into lower dimention (after tranformation v changed to ~~00~~ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$)

That means $\det(T) = 0$

area = 0

$$[A - \lambda I]\vec{v} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$\cancel{\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$\cancel{(a-\lambda)x + by} = 0$

$\cancel{x + by(d-\lambda)} = 0$

Is this eigen vectr ?

Eigen basis

If basis vector is eigen vectors

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to$ after tranformatn $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

Diagonal matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

all the basis vects and eigen vects, and diagonal entries are eigen value,

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 3x \\ 2y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3x \\ 2y \end{bmatrix} = \begin{bmatrix} 3^2 x \\ 2^2 y \end{bmatrix}$$

$$D^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} D_1^n & \cancel{0} \\ D_2^n & \cancel{0} \end{bmatrix} \begin{bmatrix} D_1^n x \\ D_2^n y \end{bmatrix}$$

~~En change the g eigen vect~~

change the basis vects so that they are the eigen vects (that we ~~we~~ calculation will be easier.)

~~A~~ change of Basis

Sandwitth original transformation with changed basis vectors and it's inverse

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

eigenvale.

Diagonalmatrix

eigen basis.

~~A~~bstract Vector Spaces

~~Determinants and eigen values~~ are indipend, of ~~choice of transform~~. coordinate.

eigen vector stay on their span

Determinats say how much transformatn — change the aree

1) Transform $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ into a unit matrix.

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 0 & 2 & -2 \end{bmatrix} \begin{matrix} R_2-2R_1 \\ R_3-3R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2/-2 \\ R_3 \to R_3-2R_2 \\ R_1 \to R_1-3R_2 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_3 \to R_3/2 \\ R_2 \to R_2+2R_3 \\ R_1 \to R_1-9R_3 \end{matrix}$$

2) Reduce the matrix into normal form and find the Rank

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 & 2 & 5/2 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & -1 & -2 & -3 \\ 0 & -7/2 & -7 & -21/2 \end{bmatrix} \begin{matrix} R_1 \to R_1/2 \\ R_2 \to R_2-3R_1 \\ R_3 \to R_3-4R_1 \\ R_4 \to R_4-9R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 \times 2 \\ R_1 \to R_1-3/2R_2 \\ R_3 \to R_3+2R_2 \\ R_4 \to R_4+7/2R_2 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} C_3 \to C_3-2C_2 \\ C_4 \to C_4-3C_2 \end{matrix}$$

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} C_3 \to C_3+C_1 \\ C_4 \to C_4-2C_2 \end{matrix}$$

Rank = 2

3) Find the Rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2/2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

Rank :- 2

4) Show that column vectors of the following matrix are linearly independent.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix} \Rightarrow$$

If linearly depended $|A| = 0$

$|A| = 1(4-3) \neq 0$ linearly independed

5) Show that vector $x_1 = (2, 4, 1, -1)$, $x_2 = (2, 3, 1, -2)$, $x_3 = (4, 6, 2, 1)$ are linearlyly independed. Express one of the vector linear combination of other.

$ax_1 + bx_2 + cx_3 = 0$ is the only ~~any~~ solution if $a = b = c = 0$ vectors are linearly Independed.

$$a \begin{bmatrix} 2 \\ 4 \\ 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix} + c \begin{bmatrix} 4 \\ 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 4 \\ 4 & 3 & 6 \\ 1 & 1 & 2 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix},$$

~~2 2 4~~

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{matrix} R_1 \to R_1/2 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - \frac{R_1}{2} \\ R_4 \to R_4 + \frac{R_1}{2} \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 \\ R_3 \to R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$a = 0$
$b + c = 0$
$c = 0$

$\Rightarrow a = 0$
$b = 0$
$c = 0$

vectors are linearly Independend.

6) Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

$$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad R_1 \to R_1 + R_2 + R_3$$

$$(a+b+c)\begin{vmatrix} 1 & & \\ 2b & b-c-a & \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$(a+b+c)(a+b+c) = (a+b+c)^2$$

---

1) The matrix $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$ is tranformed into diagonal form $D = T^{-1}AT$ where $T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Find value of θ which give the diagonal transformation,

$\cancel{A = \lambda I}$ $AX = \lambda X$

$(A - \lambda I)X = 0$

$$(A - \lambda I) = \begin{bmatrix} a-\lambda & h \\ h & (b-\lambda) \end{bmatrix}$$

$|A - \lambda I| = \cancel{(a-\lambda)^2 - h = 0}$ $(a-\lambda)(b-\lambda) = h^2$ (1)

$\cancel{a-\lambda = \pm h}$ $\cancel{\lambda^2 - \lambda a - b\lambda + ba = h^2}$

$\cancel{\lambda = a+h \text{ or } \lambda = a-h}$ $\cancel{\lambda^2 - \lambda(a+b) + ba = h^2}$

$\lambda^2 =$

$$T^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$T^{-1}AT = D \Rightarrow = a\sin\theta\cos\theta + h\cos^2\theta - h\sin^2\theta - b\sin\theta\cos\theta = 0$

$a\sin\theta\cos\theta - h\sin^2\theta + h\cos^2\theta - b\sin\theta\cos\theta = 0$

$\cancel{\lambda_1 = 0}$

$① \Rightarrow \cancel{a = \lambda_1 \text{ and } b = \lambda_2}$

$\cancel{\lambda = a \Rightarrow}$

$\cancel{\begin{bmatrix} a-a & 0 \\ 0 & b-a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0}$

$\cancel{\begin{bmatrix} 0 & 0 \\ 0 & b-a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0}$

$\cancel{(b-a)x_2 = 0}$

$\cancel{x_1 = 0 \quad = x_2 = 1}$

$\cancel{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

$\cancel{\begin{bmatrix} a-b & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0}$

$\cancel{x_1(a-b) = 0}$

$\cancel{x_2 = 0}$

$\cancel{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\cancel{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$ or $\cancel{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

~~not possible~~ ~~when $\theta = 2n\pi$~~

$$\frac{a}{2}\sin 2\theta + h\cos 2\theta - \frac{b}{2}\sin 2\theta = 0$$

① $$h\cos 2\theta = \left(\frac{b-a}{2}\right)\sin 2\theta$$

$$h = \frac{b-a}{2}\tan 2\theta$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{b-a}\right)$$

2) Find eigen vectors of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)(1-\lambda)\right] + 1\cdot(-(1-\lambda))$$

$$(1-\lambda)\left[\cancel{1}\cancel{-}\lambda - \lambda + \lambda^2 \cancel{-1}\right]$$

$$(1-\lambda)(\lambda^2 - 2\lambda) = 0$$

$$\lambda^2 - 2\lambda = 0$$

$1 - \lambda = 0$ or $\cancel{\lambda^2 = 2\lambda}$ $\lambda(\lambda - 2) = 0$

$\lambda = 1$ $\cancel{\lambda = 2}$ $\lambda = 0$ or $2 = \lambda$

for $\lambda = 1$ $(A - \lambda I)X = 0$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cancel{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$n - r$

$(3 - 2) = 1$ unk

$x_1 = 0$

$x_2 = c$

$x_3 = 0$

$$\begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} = c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 2$

$(A - \lambda I)x = 0$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(n - r) = 1$

$-x_1 + x_3 = 0$

$x_2 = 0$

$x_1 = c$

$x_3 = c$

$x_2 = 0$

eigenvector $= \begin{bmatrix} c \\ c \\ 0 \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

for $\lambda = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 + x_3 = 0$

$x_2 = 0$

$x_1 = c$

$x_3 = -c$

$x_2 = 0$

$$\begin{bmatrix} c \\ 0 \\ -c \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

eigen matrix.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$e_1 \quad e_2 \quad e_3$

$e_1^T e_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$

$e_2^T e_3 = (1 \; 0 \; 1) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 - 1 = 0$

Orthogonal

$e_1^T e_3 = (0 \; 1 \; 0) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$

3) Determine the value of α, β, and γ when the matrix

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$$ is orthogonal.

$$e_1^T e_2 = \begin{bmatrix} 0 & \alpha & \alpha \end{bmatrix} \begin{bmatrix} 2\beta \\ \beta \\ -\beta \end{bmatrix} = \alpha\beta - \alpha\beta = 0$$

$$e_2^T e_3 = \begin{bmatrix} 2\beta & \beta & -\beta \end{bmatrix} \begin{bmatrix} \gamma \\ -\gamma \\ \gamma \end{bmatrix} = 2\beta\gamma - \beta\gamma - \beta\gamma$$
$$= 2\beta\gamma - 2\beta\gamma = 0$$

$$e_1^T e_3 = \begin{bmatrix} 0 & \alpha & \alpha \end{bmatrix} \begin{bmatrix} \gamma \\ -\gamma \\ \gamma \end{bmatrix} = \gamma - \alpha\gamma + \alpha\gamma = 0$$

Column vector are orthogonal.

$$e_1^T e_1 = \begin{bmatrix} 0 & \alpha & \alpha \end{bmatrix} \begin{bmatrix} 0 \\ \alpha \\ \alpha \end{bmatrix} = \alpha^2 + \alpha^2 = 1$$
$$2\alpha^2 = 1$$
$$\alpha = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$$

$$e_2^T e_2^T = \begin{bmatrix} 2\beta & \beta & -\beta \end{bmatrix} \begin{bmatrix} 2\beta \\ \beta \\ -\beta \end{bmatrix} = 4\beta^2 + \beta^2 + \beta^2 = 1$$
$$\beta = \pm\frac{1}{\sqrt{6}}$$

$$e_3^T e_3^T = \begin{bmatrix} \gamma & -\gamma & \gamma \end{bmatrix} \begin{bmatrix} \gamma \\ -\gamma \\ \gamma \end{bmatrix} = \gamma^2 + \gamma^2 + \gamma^2 = 1$$
$$\gamma = \pm\frac{1}{\sqrt{3}}$$

Show that the vector $\begin{bmatrix} a \\ b \\ a \end{bmatrix}$ is a simutanias eisen vector of

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$Ax = \lambda x$$

$$(A - \lambda I)x = 0$$

$$A - \lambda I = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda(-(1-\lambda)\lambda) + (-1+\lambda)$$

$$= -\lambda((\lambda-1)\lambda + (\lambda-1))$$

$$= -\lambda(\lambda-1)(\lambda+1) = 0$$

$$\lambda = 0 \text{ or } \lambda = 1 \text{ or } \lambda = -1$$

$\lambda = 1$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \quad -x_1 + x_3 = 0$$

$$x_2 = c_2$$

$$x_1 = c_1$$

$$x_3 = c_1$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$Bx = \lambda x$

$(B - \lambda I)x = 0$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow$$

$(1+\lambda) - \lambda(\lambda^2 - 1) = 0$

$(1+\lambda) - \lambda(\lambda+1)(\lambda-1) = 0$

$(\lambda+1)(1 - \lambda(\lambda-1))$

$\cancel{(\lambda+1)(1+\lambda(1-\lambda))} = 0$

$(\lambda+1)(1 - \lambda^2 - \lambda)$

$1 + \lambda - \lambda^3 + \lambda$

$-\lambda^3 + 2\lambda + 1 = 0$

$\lambda^3 - 2\lambda - 1 = 0$

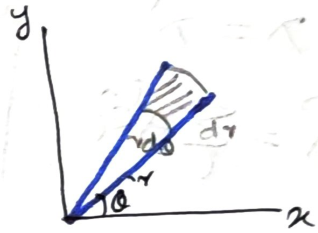
Plane polar co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2+y^2$, $\theta = \tan^{-1}(y/x)$

Note

bear in mind which quadrant the point lies

Slope in polar, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$

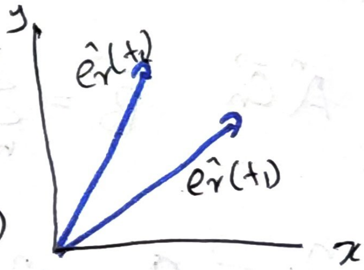


Area in polar co-ordinates,

$$\int_0^{\infty}\int_0^{2\pi} r\,dr\,d\theta$$

Vectors in plane polar coordinates

$\vec{r}(t) = r(t)\hat{e_r}$

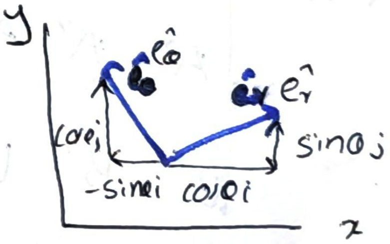


$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e_r} + r\frac{d\hat{e_r}}{dt}$$

(direction of $\hat{e_r}$ changes as t changes)
(if θ changes with time)

so $\hat{e_r} = \hat{e_r}(\theta)$ $\Rightarrow \dfrac{d\hat{e_r}}{dt} = \dfrac{d\theta}{dt}\dfrac{d\hat{e_r}}{d\theta}$

$\hat{e_r} = \cos\theta\hat{i} + \sin\theta\hat{j}$ (in cartesian)



$\dfrac{d\hat{e_r}}{d\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j} = \hat{e_\theta}$

$\perp^r$ to $\hat{e_r}$ so $\dfrac{d\hat{e_r}}{d\theta} = \hat{e_\theta}$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e} + r\frac{d\theta}{dt}\hat{e_\theta}$$

any vector $\vec{u} = u_r\hat{e_r} + u_\theta\hat{e_\theta}$, $\hat{e_r}\cdot\hat{e_\theta} = 0$

basis.

$\vec{u} = (\vec{u}\cdot\hat{e_r})\hat{e_r} + (\vec{u}\cdot\hat{e_\theta})\hat{e_\theta}$

eg: $\frac{dy}{dx} = y \Rightarrow \int \frac{1}{y} dy = \int dx$

$\log y = x$

$y = e^x$

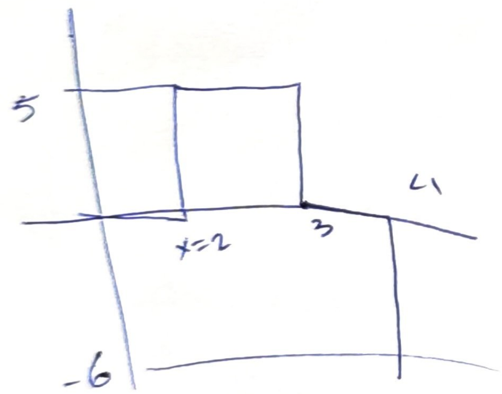
$\frac{dy}{dx} = y + \delta(x-a)$ — active at $x=a$

$\int dy = \int y \, dx + \int \delta(x-a) \, dx$

$= e^x + 1 \quad x > 1$

$y = \begin{cases} e^x, & x < a \\ e^x - 1, & x > a \end{cases}$

Heviside eqs:



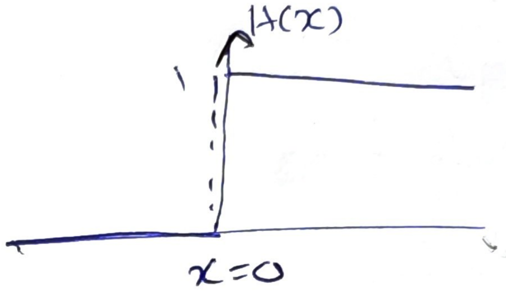
$f(x) = H(x-2)5 - 5(x-3)$

$-6H(x-4)$

Heaviside Step function / unit step function

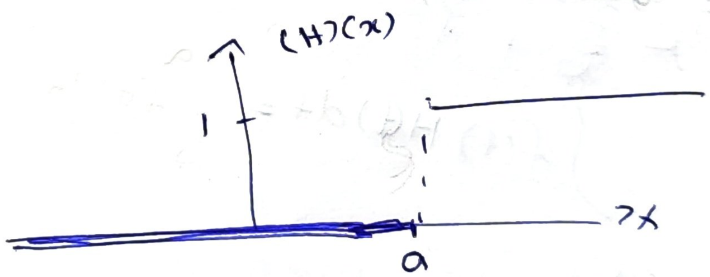
$$H(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$$

and undifind at $x = 0$



Shifted unit step function

$$\boxed{H(x-a) = \begin{cases} 0, & \text{if } x < a \\ 1, & \text{if } x > a \end{cases}}$$ and undifind at $x = a$

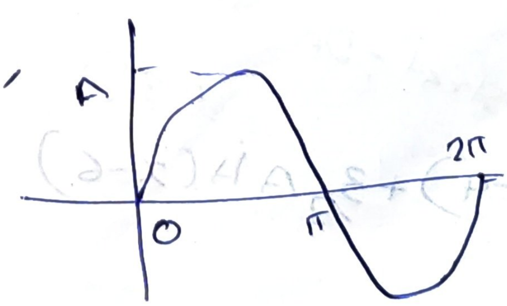


Switching on of a signal

let us consider a sinusoidal signal,

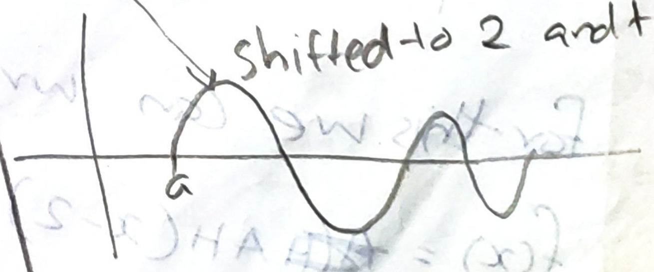
$f(x) = A\sin x$

ie,



shifted

$f(x-a)\,H(x-a)$



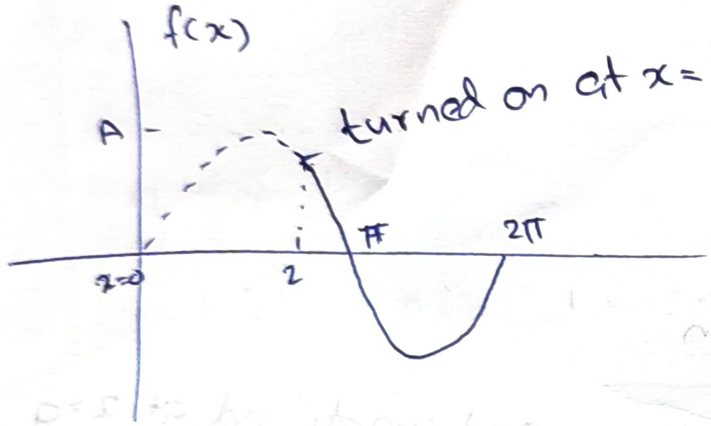
and let us consider ... we want to ... this
Signal but only $x < 2$

then we write,

$$f(x) = A\sin(x)\, H(x-2)$$

where H is heaviside function

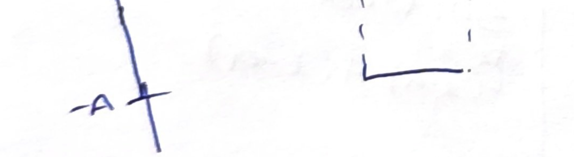
ie,



Many unit step function

let us consider that we want to denote the following signal

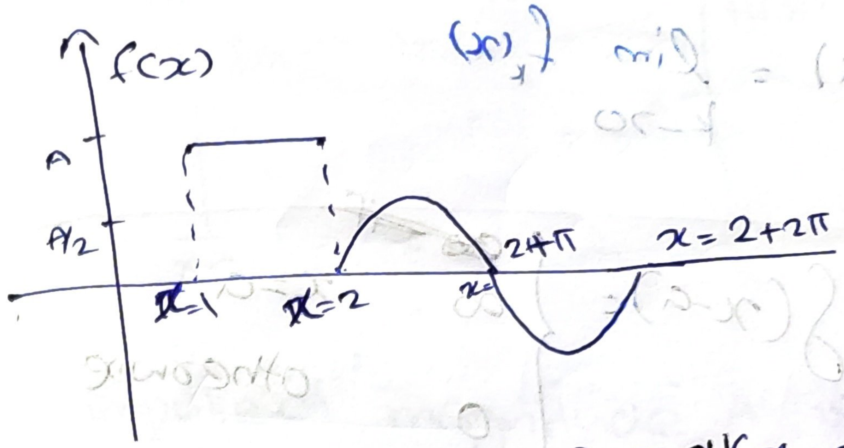
$$\int_{-\infty}^{\infty} f(t)\, H(t)\, dt = \int_{0}^{\infty} f(t)\, dt$$



for this we can write.

$$f(x) = \cancel{A} \, A\,H(x-2) \underbrace{- 2A\,H(x-4)}_{\text{substrat } -2A} + \underbrace{\tfrac{3}{2}A\,H(x-6)}_{\text{ad } 3/2A}$$

Denote the following graph using step function

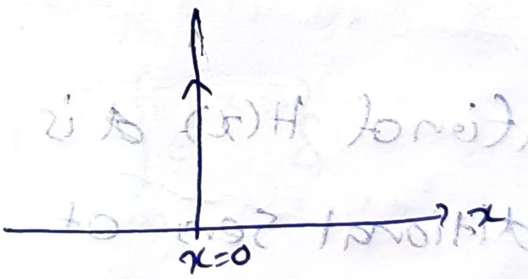


$f(x) = \cancel{A} \; A H(x-1) - AH(x-2) + \underbrace{\frac{A}{2}\sin(x-2)}_{\text{Shifted } \sin_{b_2}} H(x-2)$ starts a

Dirac delta function (unit impulse function)

Dirac deltafunction is defind as;

$$\delta(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0, & \text{if } x > 0 \\ \infty, & \text{if } x = 0 \end{cases}$$



It is also understood as at the limit of an impulse acting over a short interval at x_0

for instance

$$\cancel{f(x)}\; f_k(x-a) = \begin{cases} 0, & \text{if } x < a \\ 0, & \text{if } x > a+k \\ 1/k, & \text{if } a \le x \le \end{cases}$$

then $\delta(x-a)$ can also be ... a
the limit $$\delta(x-a) = \lim_{k\to 0} f_k(x)$$

~~Dirac delta fn~~

$$\boxed{\delta(x-a) = \begin{cases} \infty & x = a \\ 0 & \text{otherwise} \end{cases}}$$

In terms of H, $\lim_{k\to 0}$

$$f_k(x-a) = \frac{1}{k}\left[H(x-a) - H(x-(a+k))\right]$$

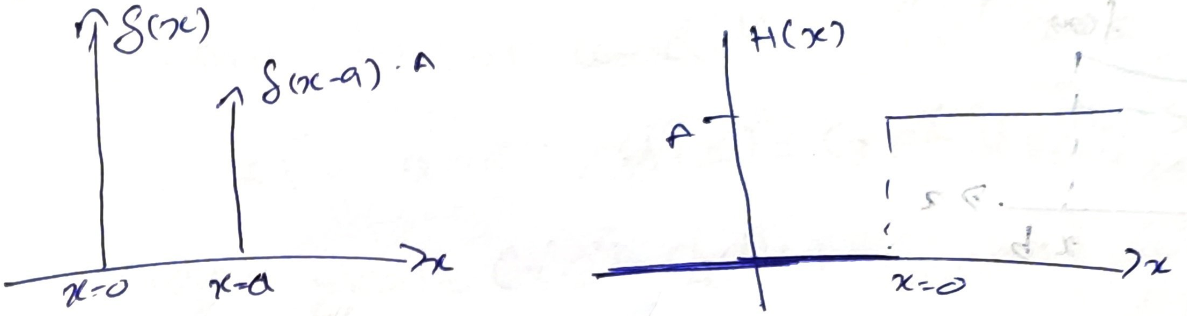
Dirac delta function can also be imagined as derivative of Heaviside step function ie,

$$\frac{dH}{dx} = \delta(x)$$

although ~~the driv~~ the derivation of $H(x)$ is not defind at $x=0$ in traditional sens of calculus. It is better if we replace the Heaviside function as integral of the delta function as,

$$\boxed{\left[H(x)\right]_{-\infty}^{\infty} = \int_{-\infty}^{\infty} \delta(x)dx = 1-0 = 1}$$

Some features of Dirac delta function



The impulse of magnitude A is applied ~~as~~ at $x=a$

then, $$\int_{-\infty}^{\infty} A\,\delta(x-a)\,dx = A\Big[H(x-a)\Big]_{-\infty}^{\infty}$$

$$= A$$

Similarly for any function

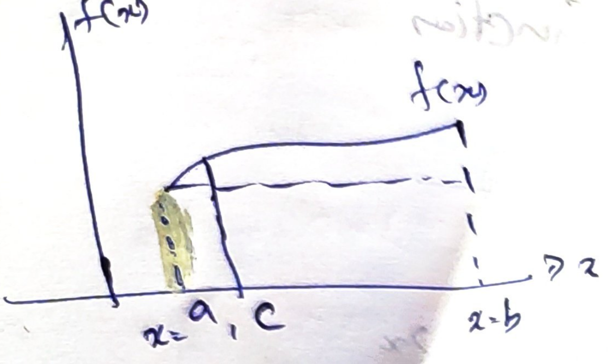
$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)}$$

eg: $$\int_{-\infty}^{\infty} e^{(x-A)}\,\delta(x-a)\,dx = e^{(a-A)}$$

Understanding ~~using mean~~ delta function using MVT

Mean Value theorm ~~state~~ says that

$$\boxed{\int_a^b f(x)\,dx = (b-a)\,f(c)}$$ where c ~~is~~ lies between a & b



Now let us apply this to integral involving $\delta(x)$

$$\int_{-\infty}^{\infty} \delta(x-a)\, f(x)\,dx$$

$$= \int_{a}^{a+k} \frac{1}{k} f(x)\,dx = (k)\,\frac{1}{k}\, f(c) = f(a)$$

vot

Examples of impulsive input:

$$\frac{dy}{dx} = y + \delta(x-a)$$

for $x < a$, $\frac{dy}{dx} = y$

$\delta(x-a) = 0$

$$\Rightarrow \frac{1}{y}\,dy = dx$$

$$\Rightarrow \log y = x + C$$

$$\Rightarrow y(x) = C_2 e^{x} \quad \text{for } x < a$$

taking exponential on both sides

$$y = e^{(x+c)}$$
$$= e^{x} \cdot e^{c} = C_2 e^{x}$$

Now at $x = a$ $\quad \frac{dy}{dx} = y + \delta(x-a)$

$\delta(x-a)$ is activated $\left(y' = y+1\right.$

$$\int \frac{y'}{y+1}\,dx = \int 1\,dx$$

$$\ln(y+1) = x + c$$

$$y = C_2 e^{x} - 1$$

$y(a) = c_2 e^a + 1$

is now the initial condition for $x > a$

now if ~~x~~ for $x > a$, $y(x) = c_3 e^x$ is the solution

then $y(a) = c_3 e^a = c_2 e^a + 1$ (initial condition).

$$\Rightarrow c_3 = \frac{c_2 e^a + 1}{e^a}$$

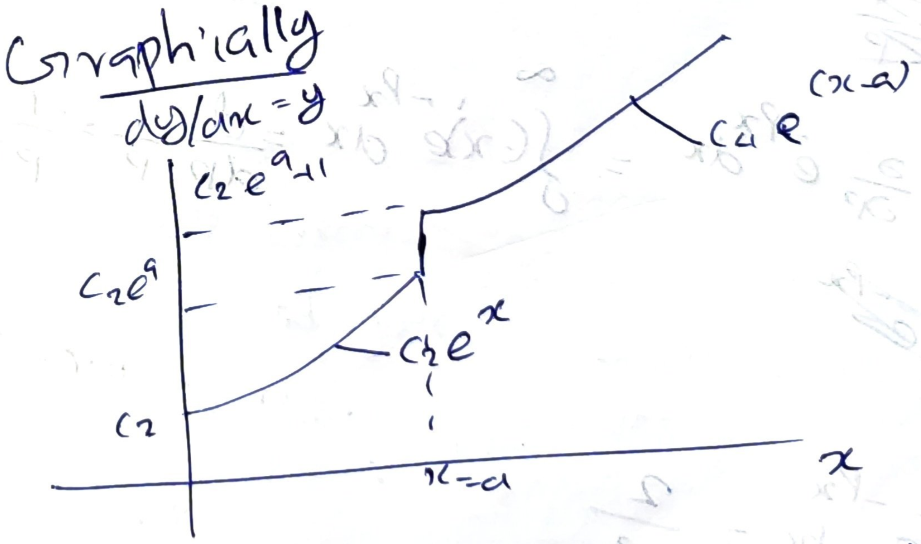
$$\& \quad y(x) = (c_2 e^a + 1) e^{(x-a)}$$

$$y(x) = c_4 e^{(x-a)} \text{ for } x > a$$

$$\text{thus } y(x) = \begin{cases} c_2 e^x & \text{for } x < a \\ c_4 e^{(x-a)} & \text{for } x > a \end{cases}$$

where $c_4 = (c_2 e^a + 1)$

Graphically



Now if initial condition is $y(0) = 0$

then $c_2 = \dfrac{y(0)}{e^0} = 0$ $\quad \& \quad c_4 = c_2 e^a + 1 = 1$

$$\text{then, } \quad y(x) = \begin{cases} 0 & \text{for } x < a \\ e^{(x-a)} & \text{for } x > a \end{cases}$$

Example of $\delta(\vec{r})$

- what is the charge dencity, when we place a charge q at, let us say, at origin?

→ charge dencity × Volume = charge

$$\int_{-\infty}^{\infty} \rho(\vec{r})\,d\vec{r} = \text{total charge}, q$$

$$\& \int_{-\infty}^{\infty} \rho(r)\,dr = q = \int_{-\infty}^{\infty} q\,\delta(\vec{r})\,d\vec{r}$$

comparis

The charge dencity is $\rho(\vec{r}) = q\,\delta(\vec{r})$

Integrating Parts

$$\left[-x^n e^{-x}\right]_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\,dx$$

$$\boxed{\left[-x^n e^{-x}\right]_0^\infty} - n x^{n-1} e^{-x}\Big]_0^\infty + \int_0^\infty n \ldots$$

$e^\infty > \infty^n$

-0

0

$$= n! \int_0^\infty e^{-x}\,dx$$

$$= n! \left(-e^{-x}\right]_0^\infty$$

$= n!$

Euler relation / Gamma function

$$\boxed{\int_0^\infty x^n e^{-x}\,dx = n!} \quad \text{—(1)}$$

Now let us consider a positive number $p > 0$ and consider rather simple integral

$$\int_0^\infty e^{-px}\,dx \quad \text{—(2)}$$

Differentiation under the integral sign ↓

(2) is simple to integrate

$$\int_0^\infty e^{-px}\,dx = -\frac{1}{p} e^{-px}\Big]_0^\infty = -\frac{1}{p}(0-1) = \frac{1}{p}$$

Now letus differentiat w.r. to p both sides :

$$\frac{d}{dp}\int_0^\infty e^{-px}\,dx = \frac{d}{dp}\frac{1}{p} = \frac{-1}{p^2}$$

$$\Rightarrow \int_0^\infty \frac{\partial}{\partial p} e^{-px}\,dx = \int_0^\infty -xe^{-px}\,dx = \cancel{\frac{1}{p}} -\frac{1}{p^2}$$

$n(n-1)x^{n-2}$

$e^{-x}dx$

differentiate again

$$\frac{d}{dp}\int_0^\infty -xe^{-px}\,dx = \frac{d}{dp}\left(-\frac{1}{p^2}\right)$$

$$\Rightarrow \int_0^\infty \frac{\partial}{\partial p} -xe^{-px}\,dx = \int_0^\infty (-x)^2 e^{-px}\,dx = \frac{2}{p^3}$$

If we follow theis seqaencal differentia by $\frac{d^n}{dp^n}$, the,

$$\boxed{\int_0^\infty (-x)^n e^{-px}\,dx = (-1)^n \frac{n!}{p^{(n+1)}}}$$

this we can also write

$$\boxed{\int_0^\infty \cancel{(-1)^n} x^n e^{-px}\,dx = \cancel{(-1)^n}\frac{n!}{p^{(n+1)}}}$$

Now we set $p=1$

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

Due to some reson in literature we usually ; shifted factorial

$$\boxed{\int_0^\infty x^{(n-1)} e^{-x}\, dx = (n-1)! = \Gamma(n)}$$

This is the gamma f^n

Properties of Gama function

we defind and proved that gamma functin $\Gamma(n)$ is

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

then $$\Gamma(n+1) = \int_0^\infty \underbrace{x^n}_{\text{I}} \underbrace{e^{-x}}_{\text{II}}\, dx$$

Integrating by parts:

$$\Gamma(n+1) = \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty n x^{n-1} (-e^{-x})\, dx$$

$$= 0 + n \underbrace{\int_0^\infty x^{(n-1)} e^{-x}\, dx}_{\Gamma(n)}$$

$$\boxed{\Gamma(n+1) = n\Gamma(n)}$$

Now, let us say $n=1$ hence

$\Gamma(2) = \Gamma(1) =$

if $(n=2)$, $\Gamma(3) = 2\Gamma(2) = 2\Gamma(1)$

if $(n=3)$, $\Gamma(4) = 3\Gamma(3) = 3\times2\times\Gamma(2) = 3\times2\times1\Gamma($

if n $\boxed{= \Gamma(n+1) = n\Gamma(n) = n!}$

$$n! = \int_0^\infty x^n e^{-x}\,dx$$

$$\Gamma(n) = (n-1)! \Rightarrow n! = \Gamma(n+$$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$$

$$\Gamma(n) = n\Gamma(n-1) \Rightarrow \Gamma(n+1) =$$

$$\Gamma(1) = 1$$

→ what is $\Gamma(1) = ?$

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x}\,dx$$

$$= \int_0^\infty e^{-x}\,dx$$

$$= \left(-e^{-x}\right]_0^\infty$$

$$\Gamma(1) = 0 - (-1) = \underline{1}$$

In complete gamma function

Breaking the integral.

$$\Gamma(n) = \int_0^\infty x^{(n-1)} e^{-x}\,dx = \int_0^t x^{(n-1)} e^{-x}\,dx + \int_t^\infty x^{(n-1)} e$$

$$\Gamma(n) = P(n,t) + R(n,t)$$

P and R are called incomplete gamma function

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{(1/2-1)} dx$$

$$= \int_0^\infty e^{-x} x^{-1/2} dx$$

$$= \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx$$

substituting $\sqrt{x} = t \to x = t^2$

$dx = 2t\,dt$

and limits unchange.

$$\Rightarrow \Gamma(1/2) = \int_0^\infty e^{-t^2} \cancel{t^{-1}}\, 2\cancel{t}\, dt$$

$$= 2\int_0^\infty e^{-t^2} dt$$

$$\Gamma(1/2) = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$$

Gaussian integral.

$\int_{-\infty}^{\infty} e^{-x^2} dx = I$, $\int_{-\infty}^{\infty} e^{-y^2} dy = I$

$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx\,dy$ in polar $-(x^2+y^2)$

$= \int_0^\infty \int_0^{2\pi} e^{-r^2} r\,dr\,d\theta$, $u = e^{-r^2}$, $du = \frac{e^{-r^2}}{2}$

$2\pi \times \int_1^0 \frac{du}{2} = 2\pi \times \frac{1}{2} = \pi =$

$I = \sqrt{\pi}$

$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$

by symmetry.

Problem: ~~Elaburate~~ valuate, $\int_0^1 (x \log x)^4 dx$

$\log x = -t$

$x = e^{-t}$

$dx = -e^{-t} dt$

limits, $x = 0 \Rightarrow t = \infty$ sus

$x = 1 \Rightarrow t = 0$

$$I = \int_{-\infty} \left[e^{-t}(-t)\right]^4 (-e^{-t})\, dt$$

$$= -\int_{-\infty} e^{-5t} t^4 \, dt$$

$$= +\int_0^{\infty} e^{-5t} t^4 \, dt$$

If limit is 0 to ∞

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\, dy = \_\_\_$$

$$\int_0^\infty \int_0^{\pi/2} e^{-r^2} r\, dr\, d\theta$$

$$\frac{\pi}{2} \times \frac{1}{2}$$

in QM

$\int_{-\infty}^{\infty} e^{-x^2/5} dx =$ put $x' = \frac{x}{\sqrt{5}}$

$x'^2 =$

$$\int_{-\infty}^{\infty} e^{-x'^2} \frac{dx'}{\sqrt{5}}$$

$\Rightarrow \sqrt{5}\sqrt{\pi}$

Now set $5t = y \Rightarrow t = y/5$

$\Rightarrow dt = \frac{dy}{5}$ and limits un changed

$$\Rightarrow I = \int_0^\infty e^{-y} (y/5)^4 \frac{dy}{5}$$

$$I = \frac{1}{5^5} \int_0^\infty e^{-y} y^4 \, dy = \frac{1}{5^5} \Gamma(5)$$

Gamma function for $n > 0$

$\Gamma(1/2) = \sqrt{\pi}$

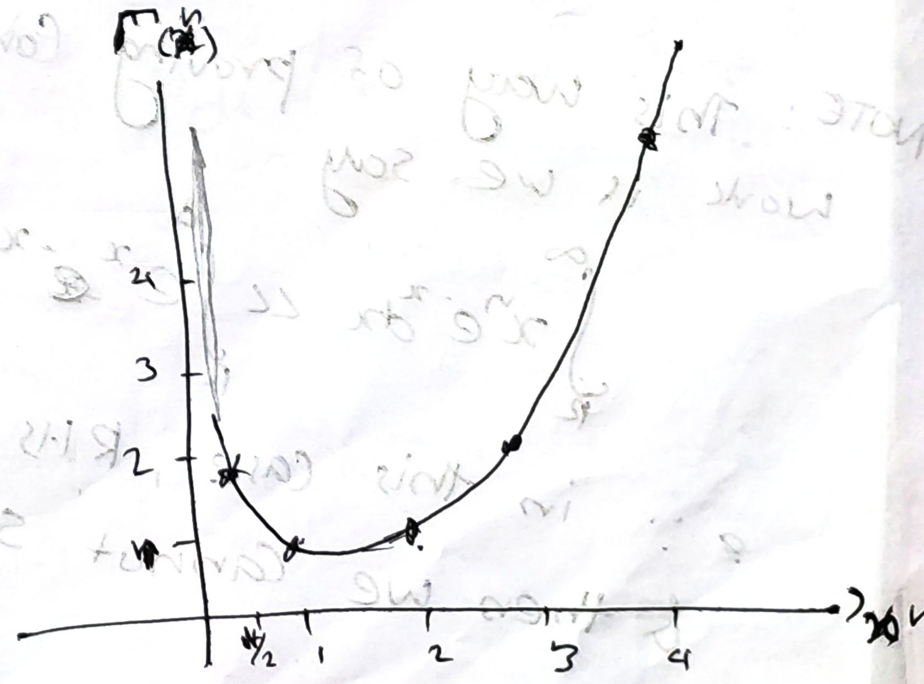
$\Gamma(1) = 1$

$\Gamma(2) = 1$

$\Gamma(3) = 2$

$\Gamma(4) = 6$

⋮



→ ~~If~~ is $\int_0^{\infty} x^n e^{-x} dx$ Convegent?

$$\int_R^{\infty} x^n e^{-x} dx << \underbrace{\int_R^{\infty} e^{-x/2} e^{-x} dx}$$ for lagerR

$$= \int_R^{\infty} e^{-x/2} dx$$

$$= -2e^{-x/2}\Big]_R^{\infty}$$

$$= 2e^{-R/2}$$

IF $R=0$ RHS $= 2$

and LHS $= 1$ (How) (because!! =1)

This if RHS is converging, and LHS is smaller than RHS the LHS is also conversing.

NOTE: This way of proving Convergence would not work is we say

$$\int_R^{\infty} x^n e^{-x} dx << \int_R^{\infty} e^{x} \cancel{e}^{-x} dx$$ for R larger

in this case, RHS is infinite.

& then we cannot say that LHS is finite.

$$\boxed{\lim_{n\to\infty} \frac{n!}{f(n)} = 1}$$ means $f(n)$ is asymptotic representation of $n!$

→ some function

It is also called "asymptotic expantion" (Stirling formulae)

→ error between them gets smaller & smaller

$$n! = \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt$$

$$t^n = z$$

$$n \log t = \log z$$

$$e^{\log(t^n)} = e^{\log z}$$

or $\log t = \frac{1}{n} \log z$

or $t^n = \left(e^{1/n \log z}\right)^n$

$$t^n = e^{\log z}$$

$$n! = \Gamma(n+1) = \int_0^\infty e^{\log z} e^{-t} dt$$

$$= \int_0^\infty e^{(\log z - t)} dt$$

$$= \int_0^\infty e^{(n \log t - t)} dt$$

follow ⇓

$$n! = \int x^n e^{-x} dx$$

$$= \int t^n e^{-t} dt$$

$$= \int e^{\log(t^n)} e^{-t} dt$$

$$= \int e^{n \log t - t} dt$$

nice aproximation for log(t)

→ we want to factor out n also

Substitute: limits

$t = n(1+x)$

$dt = n\,dx$

limits:

t	0	∞
x	-1	∞

$$\Rightarrow n! = \Gamma(n+1) = \int_{-1}^{\infty} n\,dx \cdot e^{n\log[n(1+x)] - n(1+x)}$$

$$= \int_{-1}^{\infty} n\,dx \cdot e^{n\log n + n\log(1+x) - n - nx}$$

$$= n e^{\underbrace{n\log n}_{\log(n^n)}} e^{-n} \int_{-1}^{\infty} e^{n\log(1+x) - nx}\,dx$$

$$= n n^n e^{-n} \int_{-1}^{\infty} e^{n[\log(1+x) - x]}\,dx$$

$$\boxed{n! \approx n^{n+1} e^{-n} \int_{-1}^{\infty} e^{n(\log(1+x) - x]}\,dx}$$

convert this asymptotically into another integral

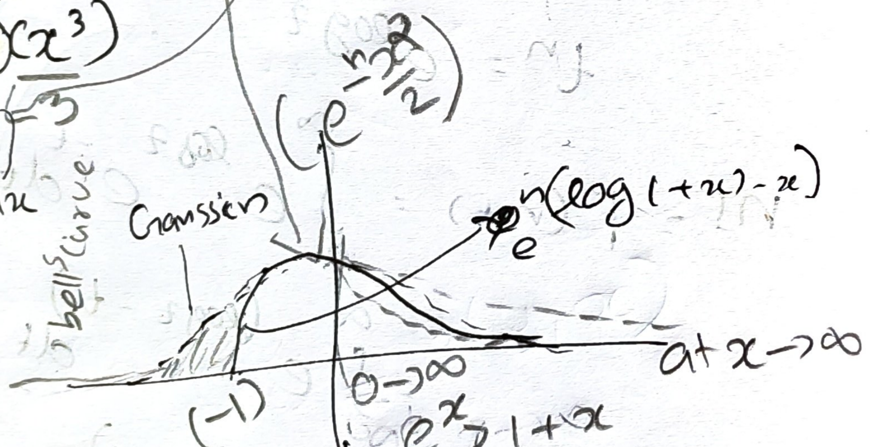
$$\log(1+x) = x - \frac{x^2}{2} + O(x^3)$$ (3)

then $\int_{-1}^{\infty} e^{n(\cancel{x} - x^2/2 - \cancel{x})}\,dx$

$$= \int_{-1}^{\infty} e^{-nx^2/2}$$

$$\approx \int_{-\infty}^{\infty} e^{-nx^2/2}\,dx$$

(because the error will be small
easier to compute



$e^x > 1+x$

$x > \ln(1+x)$

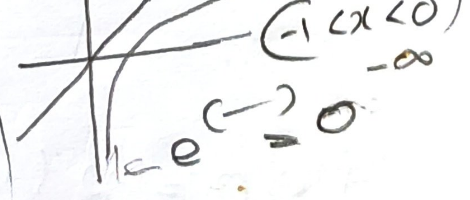
$x - x > \ln(1+x) - x$

$0 > -\infty$

$e^{(-)} \to 0$

at $x \to 0$

$(-1 - 0)$ lower the



Substitute : $+n\frac{x^2}{2} = z^2$

get gaussian integral

$$\Rightarrow \sqrt{n/2}\,x = z$$

$$\Rightarrow \sqrt{\frac{n}{2}}\,dx = dz$$

$$\int_{-\infty}^{\infty} e^{-z^2}\sqrt{\frac{2}{n}}\,dz = \sqrt{\frac{2}{n}}\sqrt{\pi}$$

$$\sqrt{\frac{2\pi}{n}}$$

$\therefore n! \approx n^{n+1} e^{-n} \sqrt{\frac{2\pi}{n}}$ — v

error!! =

10! = ·

or $\boxed{n! \approx n^n e^{-n} \sqrt{2\pi n}} = \Gamma(n+1)$

using Stirling's aproximation, large factorials can be calculated.

Relation between Gamma function & Beta function

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-t} t^{x-1} dt \int_0^\infty e^{-u} u^{y-1} du$$

$$= \int_0^\infty\int_0^\infty e^{-(t+u)} t^{x-1} u^{y-1} dt du$$

mutivariable calculus

now set $t+u = v$

also $t+u = v(1-\tau+\tau)$

$t+u = v(1-\tau) + v\tau$

$0 < t < \infty$
$0 < u < \infty$
$\Rightarrow 0 < v < \infty$
$0 < \tau < \infty$
$(1-\tau) > 0$ $\quad 1 > \tau$

comparing

$t = v(1-\tau)$
$u = v\tau$

limits

t	0	∞
$v(1-\tau)$	0	∞
τ	0	1

$v > 0$

u	0	∞
$v\tau$	0	∞
v	0	∞

& $\tau > 0$

then $\Gamma(x)\Gamma(y) = \int_0^1\int_0^\infty e^{-v} [v(1-\tau)]^{x-1} (v\tau)^{y-1} d\tau dv$

Jacobian determinant order doesn't matter

$$|J| = \begin{vmatrix} \frac{\partial u}{\partial \tau} & \frac{\partial u}{\partial v} \\ \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} v & \tau \\ -v & 1-\tau \end{vmatrix}$$

$= v(1-\tau) + v\tau$

$= v - \cancel{v\tau} + \cancel{v\tau}$

$J = v$

$$\Rightarrow \Gamma(x)\Gamma(y) = \int_0^1 \int_0^\infty e^{-v} v^{x-1} (1-\tau)^{x-1} v^{y-1} \tau^{y-1} v \, d\tau \, dv$$

$0 < v < \infty$
$0 < \tau < 1$

$$= \int_0^\infty v^{(x+y-1)} e^{-v} dv \int_0^1 (1-\tau)^{x-1} \tau^{y-1} d\tau$$

$$\Rightarrow \Gamma(x)\Gamma(y) = \Gamma(x+y) \int_0^1 \tau^{y-1} (1-\tau)^{x-1} d\tau$$

$$\Rightarrow \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 \tau^{y-1} (1-\tau)^{x-1} d\tau$$

$$= \boxed{\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \beta(x,y)} = \beta(y,x)$$

symmetric

or we show that

$$\frac{(x-1)!\,(y-1)!}{(x+y-1)!} = \boxed{\int_0^1 \tau^{y-1} (1-\tau)^{x-1} d\tau}$$

or $$\frac{x!\,y!}{(x+y+1)!} = \int_0 \tau^y (1-\tau)^x d\tau$$

Example

$$B(2,3) = \int_0^1 x^{2-1}(1-x)^{3-1}\,dx$$

$$= \int_0^1 x(1-x)^2\,dx$$

$$= \int_0^1 x(1+x^2-2x)\,dx$$

$$= \int_0^1 (x+x^3-2x^2)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left(\frac{x^4}{4}\right]_0^1 - 2\left[\frac{x^3}{3}\right]_0^1$$

$$= 1/2 + 1/4 - 2(1/3)$$

$$= 1/2 + 1/4 - 2/3$$

$$B(2,3) = \frac{6+3-8}{12} = 1/12$$

also from $B(2,3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)}$

$$= \frac{1!\ 2!}{4!} = \frac{2}{24} = \frac{1}{12}$$

→ Symmetry Properties of B function $\frac{x!\,y!}{(x+y+1)!} =$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

Put $1-t = \tau$

$-dt = d\tau$

t	0	1
τ	1	0

$$\beta(x,y) = -\int_1^0 (1-\tau)^{x-1} \tau^{y-1} d\tau$$

$$= \int_0^1 \tau^{y-1}(1-\tau)^{x-1} d\tau$$

$$= \beta(y,x)$$

$\therefore$ β function is <u>Symmetric</u>

<u>Trigonometric form of β function</u>

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$$

if $x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

limit

x	0	1
θ	0	$\pi/2$

$$\Rightarrow \beta(a,b) = \int_0^{\pi/2} (\sin^2\theta)^{a-1}(\cos^2\theta)^{b-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2a-2+1}(\cos\theta)^{2b-2+1} d\theta$$

$$\boxed{\beta(a,b) = 2\int_0^{\pi/2} (\sin\theta)^{2a-1}(\cos\theta)^{2b-1} d\theta}$$

eg: $\beta(1,1) = 2\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$ | $\beta(2,2) = 2\int_0^{\pi/2} \sin^3\theta\cos^3\theta\, d\theta$

etc

Introduction to tensors

$$H(x-a) = \begin{cases} 0 ; & x<a \\ 1 ; & x>a \end{cases}$$ $x=a$ undefin

$$\int_{-\infty}^{\infty} f(t) H(t) dt = \int_0^t f(t) dt$$

$(t-a)$ $= \int_a^t f(t) \ldots$

$$\delta(x-a) = \begin{cases} 0 ; & x<a \\ 0 ; & x>a \\ \infty ; & x=a \end{cases}$$

$\delta = \frac{dH}{dx}$

$$\int_{-\infty}^{\infty} f(x) \delta(x-a) = f(a)$$

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\right.$$

$$\Gamma(n) = (n-1)! = \int_0^{\infty} x^{(n-1)} e^{-x} dx$$

recursion relation

$$\Gamma(n+1) = n\Gamma(n) =$$ $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$

for complex

stilings apro

$$n! = n^n e^{-n} \sqrt{2\pi n}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$ is symmetric $\Rightarrow B(y,x)$

$$B(x,y) = \frac{(x-1)!\,(y-1)!}{(x+y-1)!} = \int_0^1 \tau^{y-1} (1-\tau)^{x-1} d\tau$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1}$$

$$= \int_0^{\infty} \frac{\tau^{x-1}}{(1+\tau)^{x+y}} d\tau$$

$\vec{\nabla} \cdot \vec{u} = $ flow

$\vec{\nabla} \times \vec{u} = $ rotation.

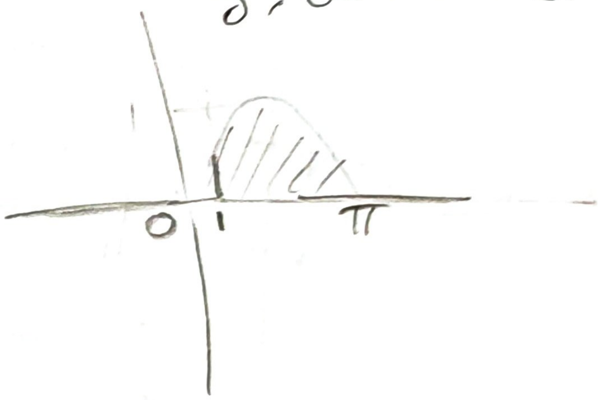
$\vec{\nabla}(\vec{u}) = $ charge.

$$\vec{\nabla} \vec{u} = \begin{bmatrix} \frac{\partial}{\partial x} \hat{i} \\ \partial y \, \hat{j} \end{bmatrix} \begin{bmatrix} u_x & u_y \end{bmatrix}$$

1) Calculate the following integral and show the results graphically

Θ, step function
δ, dirac delta

a) $\int_0^{\pi} \sin(x)\,\Theta(x-1)$

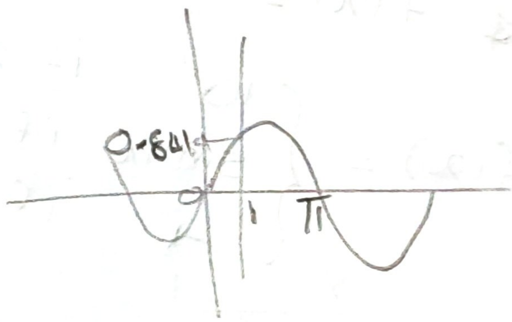


$= \int_1^{\pi} \sin x\,dx = -[\cos x]_1^{\pi}$

$= -\cos\pi + \cos(1)$

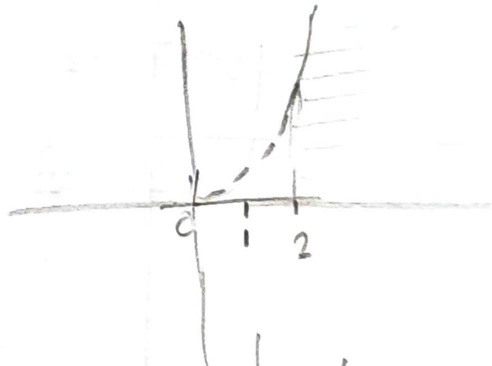
$= 1.5403$

b) $\int_0^{\pi} \sin(x)\,\delta(x-1)\,dx$



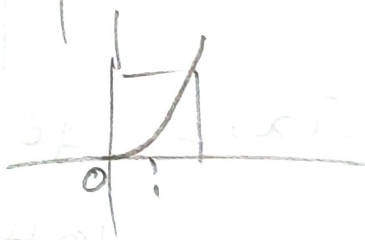
$= \sin(1) = 0.841$

c) $\int_0^{1} x^2\,\Theta(x-2)\,dx$



$= 0$

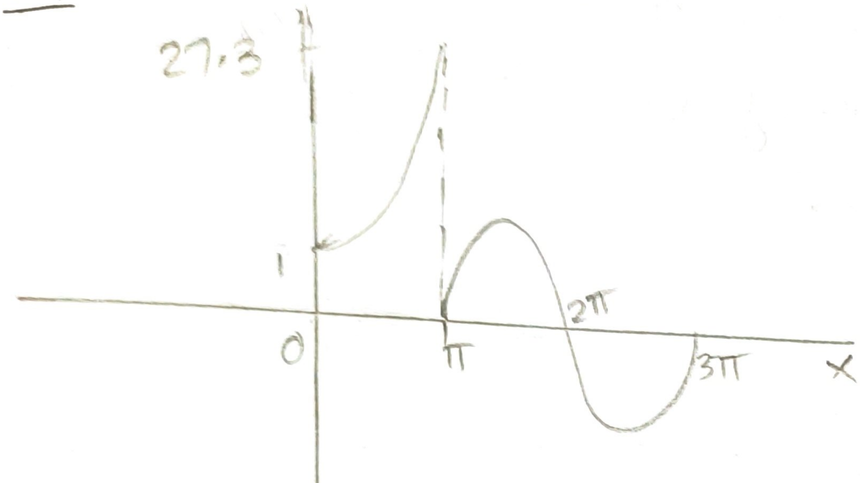
d) $\int_0^{1} x^2\,\delta(x-2)\,dx$



$= 0$

2) Represent the following result using step function and illustrate your result graphically

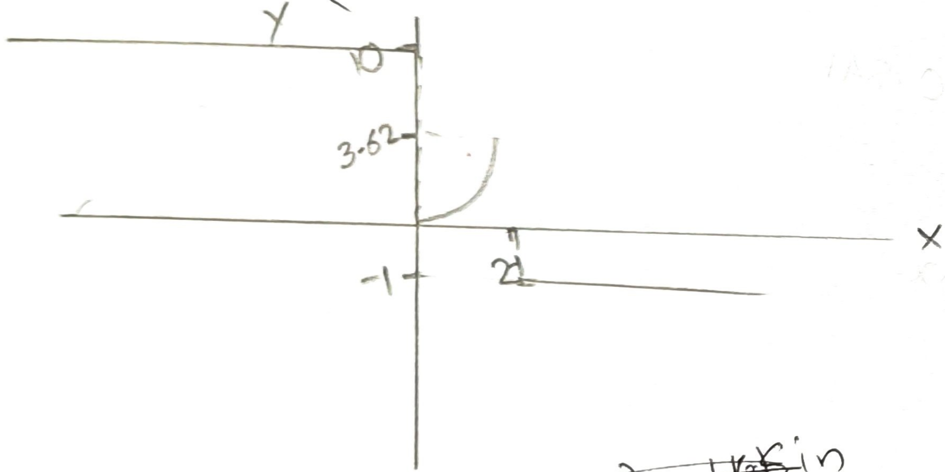
a) a> $f(x) = \begin{cases} 0 & \text{if } x<0 \\ e^x & \text{if } 0<x<\pi \\ \sin(x) & \text{if } x\geq\pi \end{cases}$



$$e^{x}H(x) - 27.3H(x-\pi) + \sin(x-\pi)H(x-\pi)$$

$$f(x) = \cancel{H(x)} + \cancel{e^{x}H(x) + \sin(x-\pi)H(x-\pi)}$$

b) $$f(x) = \begin{cases} 10 & \text{if } x<10 \\ \sinh x & \text{if } 0 \le x < 2 \\ -1 & \text{if } x \geq 12 \end{cases}$$



$$f(x) = \cancel{10H(x+\infty) + H(x)\sin}$$

$$= \cancel{10H(x+\infty) + H(x)\sinh x - 1H(x-2)}$$

$$= 10H(x+\infty) - 10H(x) + H(x)\sinh x - 4.62H(x-2)$$

3) Show that $\frac{\Gamma(x)}{\Gamma(x+1)}$ is an asymptote to $\frac{e}{(x-1)}$

In the limit $x \to \infty$

Ans)

Asymplotic expantion of $\Gamma(n+1)$

Stirlings aproximation, $n! = \Gamma(n+1) = n^n e^{-n} \sqrt{2n\pi}$

$$\Gamma(x) = (x-1)! \approx (x-1)^{(x-1)} e^{-(x-1)} \sqrt{2(x-1)\pi} \quad -①$$

$$\Gamma(x+1) = x! \approx x^x e^{-x} \sqrt{2\pi x} \quad -②$$

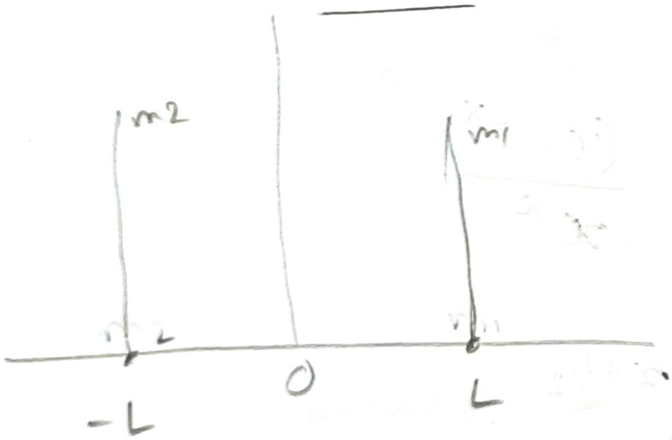
$$\frac{\Gamma(x)}{\Gamma(x+1)} = \frac{①}{②} \Rightarrow \frac{(x-1)^x}{(x-1)\, x^x} e^{-\cancel{x}+1+\cancel{x}} \frac{\sqrt{(x-1)}}{\sqrt{x}}$$

$$= \frac{e}{(x-1)} \left(\frac{(x-1)}{x}\right)^{1/2} \frac{(x-1)^x}{x^x}$$

$$= \frac{e}{(x-1)} \left(\frac{x-1}{x}\right)^{x+1/2}$$

4) Two point masses m1 and m2 placed at $x = L$ & $x = -L$ respectively represent a mass dencity field $\rho(x)$ using Dirac delta function and show the obvious result that total mass $M = m_1 + m_2$ is equals to $\int \rho(x)\,dx$ Now in $\rho(x)$ place the dirac delta function $\delta(x)$ with a fuction $g(x) = \left(\frac{1}{\sqrt{\pi}\,h}\right) e^{-x^2/h^2}$, which is in the limit $h \to 0$ aproches $\delta(x)$ consider a special typeof particle which tednds to move towards minimum of potential $V(x) = \nabla \rho(x)$ find that whether such minimum of $v(x)$ exist, and if yes, the at which location ? also plot $\rho(x)$ and $v(x)$ and discuss the result in limit $h(x) \to \infty$

Ans



$m_1(x) = m_1 \delta(x - L)$

$m_2(x) = m_2 \delta(x + L)$

i) $\rho(x) = m_1 \delta(x - L) + m_2 \delta(x + L)$

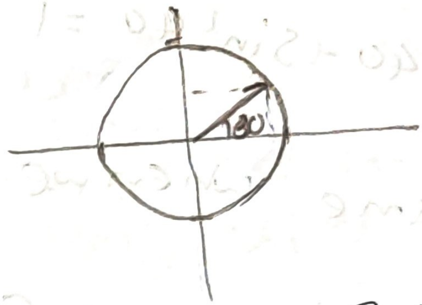
ii) $M = \int \rho(x)\,dx = \int m_1 \delta(x - L)\,dx + \int m_2 \delta(x + L)\,dx$

$\int \delta(x)\,dx = 1$

$= m_1 + m_2$

$M = m_1 + m_2$

5) unit vector a was attached to a rod in 2D spa[ce]
consider different allignment of rod/unit vect[or]
with horizotel axis. ~~30~~ 30°, 120°, 320° find
out
a) the tensor $\overline{\overline{T}} = n \times n$ in each of the case



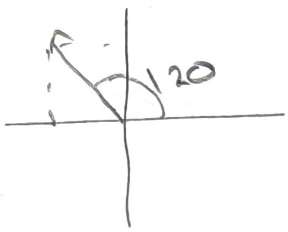
$x = \cos 30^\circ = \frac{\sqrt{3}}{2}$

$y = \sin 30 = \frac{1}{2}$

$\hat{n} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$

$\hat{n} \otimes \hat{n} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 \end{bmatrix}$

$\overline{\overline{A}} = \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix}$

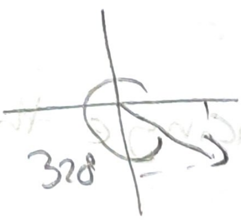


$x = -\cos 60 = -\frac{1}{2}$

$y = \sin 60 = \sqrt{3}/2$

$\hat{n} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$

$\hat{n} \otimes \hat{n} = \overline{\overline{B}} = \begin{bmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{bmatrix}$



$x = \cos 40 = 0.767$

$y = -\sin 40 = -0.64$

$\hat{n} = \begin{bmatrix} 0.77 \\ -0.64 \end{bmatrix}$

$n \otimes n = \overline{\overline{C}} = \begin{bmatrix} 0.59 & -0.49 \\ -0.49 & 0.43 \end{bmatrix}$

b) Find out trace in ~~ea~~ each case. Now trace change when we change allignment.

$tr(\bar{\bar{A}}) = \frac{4}{4} = 1$

$tr(\bar{\bar{B}}) = 1$

$trac(\bar{\bar{C}}) = \cos^2 40 + \sin^2 40 = 1$

Trace are same when we change the allignment

c) Find det$(\bar{\bar{T}})$ in each case and how det$(\bar{\bar{T}})$ change in in each case

$det(\bar{\bar{A}}) = \frac{3}{16} - \frac{3}{16} = \cancel{\frac{9}{16}}\ 0$

$det(\bar{\bar{B}}) = \frac{3}{16} - \frac{3}{16} = 0$

$det(\bar{\bar{C}}) = \cos^2 40 \cdot \sin^2 40 - (\cos 40 \sin 40)^2$

$= \cancel{\cos^2 40 \cos}\ 0$

Determinent is zero when we change the allignment.

6) ~~f(x)~~ $= f(x_1, x_2, x_3) = e^{x_1 x_2 x_3}$ write down the matrix representation of following tensor.

$\overline{\overline{T}} = T_{ij} = \frac{\partial^2 f}{\partial x_i x_j}$; $i,j \in \{1,2,3\}$

Did you find $T_{ij} = T_{ji} = ?$ Now consider a vec $\vec{n} = 2\hat{i} - \hat{j} + 4\hat{k}$, $\hat{i}, \hat{j}, \hat{k}$ are unit vectors in the direction ~~eton~~ along x_1, x_2, x_3 respectively. Find $\vec{n} \cdot \overline{\overline{T}}$ and $\overline{\overline{T}} \cdot \vec{n}$ is $n \cdot \overline{\overline{T}} = \overline{\overline{T}} \cdot \hat{n}$ If yes is th resut universal?

$\frac{\partial f}{\partial x_1} = e^{x_1 x_2 x_3} \times x_2 x_3$; $\frac{\partial^2 f}{\partial x_1^2} = e^{x_1 x_2 x_3} (x_2 x$

$= x_2^2 x_3^2 e^{x_1 x_2 x_3}$

$\frac{\partial f}{\partial x_1 x_2} = x_1 x_2 x_3^2 e^{x_1 x_2 x_3} + x_3 e^{x_1 x_2 x_3} = x_3 e^{x_1 x_2 x_3} (x_1$

$\frac{\partial f}{\partial x_1 x_3} = x_2 e^{x_1 x_2 x_3} (x_1 x_2 x_3 + 1)$

$\frac{\partial^2 f}{\partial x_2^2} = x_1^2 x_3^2 e^{x_1 x_2 x_3}$

$\frac{\partial^2 f}{\partial x_2 x_3} = x_1 e^{x_1 x_2 x_3} (x_1 x_2 x_3 + 1)$

$\frac{\partial^2 f}{\partial x_2 x_1} = x_3 e^{x_1 x_2 x_3} (x_1 x_2 x_3 + 1)$

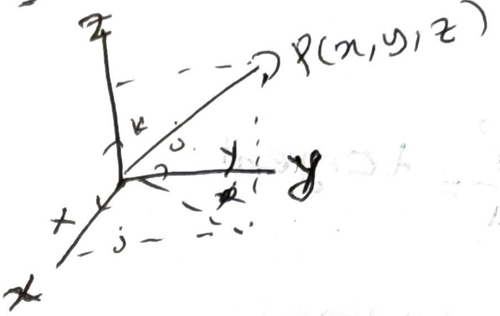
$\frac{\partial^2 f}{\partial x_3 x_1} = x_2 e^{x_1 x_2 x_3} (x_1 x_2 x_3 + 1)$

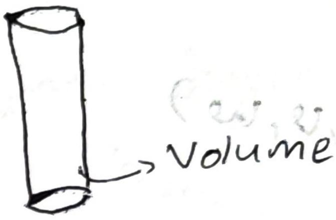
$$① \Rightarrow a^2+34=1$$
$$a^2=-33$$
$$a=\pm\sqrt{33}\,i$$

$$② \Rightarrow \pm\sqrt{28}\,i$$

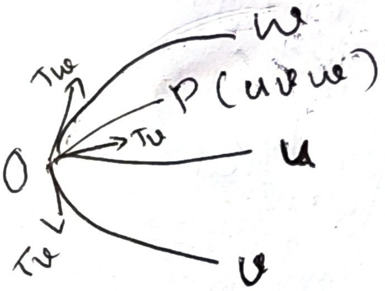
Curvelinear Co-ordinates

Cartetion Coordinates





refer back



component of r along x

function of

$$x = X(u, v, w)$$

$$y = Y(u, v, w)$$

$$z = Z(u, v, w)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = u\hat{T}_u + v\hat{T}_v + w\hat{T}_w$$

$$u = U(x, y, z)$$

$$v = V(x, y, z)$$

$$w = W(x, y, z)$$

Orthogonal curvelinear co-ordinate

$$\hat{T}_u \cdot \hat{T}_v = \hat{T}_u \cdot \hat{T}_w = 0 = \vec{T}_v \cdot \vec{T}_w = 0$$

$$|\hat{T}_u \times \hat{T}_v| = 1$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= x(u,v,w)\,i + y(u,v,w)\,j + z(u,v,w)\,\hat{z}$$

$$= \vec{r}(u,v,w)$$

tangent to the curve.

$\frac{\partial \vec{r}}{\partial u}$ = tangent

$$\boxed{\hat{T}_u = \frac{\frac{\partial \vec{r}}{\partial u}}{\left|\frac{\partial \vec{r}}{\partial u}\right|}}$$

tangent

scale factor

$$\hat{T}_w = \frac{\frac{\partial \vec{r}}{\partial w}}{\left|\frac{\partial \vec{r}}{\partial w}\right|}$$

$$\hat{T}_v = \frac{\frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial v}\right|}$$

$$h_1 = \left|\frac{\partial \vec{r}}{\partial u}\right|,\quad h_2 = \left|\frac{\partial \vec{r}}{\partial v}\right|,\quad h_3 = \left|\frac{\partial \vec{r}}{\partial w}\right|$$

for ~~co-ordination~~ cartition coordinate $h = 1$

In Polar

$\vec{r} = x\hat{i} + y\hat{j}$

$= r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

$h_1 = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = r$

$$\boxed{d\vec{r}(u,v,w) = \frac{\partial \vec{r}}{\partial u}du + \frac{\partial \vec{r}}{\partial v}dv + \frac{\partial \vec{r}}{\partial w}dw}$$

$$= h_1 \hat{T}_u du + h_2 \hat{T}_v dv + h_3 \hat{T}_w dw$$

$$\boxed{ds^2 = d\vec{r}\cdot d\vec{r}}$$

$$ds^2 = h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2$$

Polar,

$ds^2 = dr^2 + r^2 d\theta^2$

$$ds = \sqrt{h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2}$$

$ds_1 = h_1 du$. $ds_2 = h_2 dv$ $ds_3 = h_3 dw$

$dV(u_1 u_2 u_3) = a_{>1}\, a_{>2}\, a_3$
$= h_1 h_2 h_3\, du\, dv\, dw$

Vector operation In Curvelinear Coordinates

* Gradient,

In cartetion $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

let f be any scalar function in curvelinear coordinate.

aim $\vec{\nabla} f$

$r(u, v, w)$

$d\vec{r} = \frac{\partial \vec{r}}{\partial u}du + \frac{\partial \vec{r}}{\partial v}dv + \frac{\partial \vec{r}}{\partial w}dw$

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw \quad -①$$

$= \nabla f \cdot d\vec{r}$

$$\boxed{d\vec{r} = h_1 du\,\hat{i}_u + h_2 dv\,\hat{i}_v + h_3 dw\,\hat{i}_w} \quad -ⓑ$$

$$df = (\vec{\nabla} f)_u h_1 du + (\vec{\nabla} f)_v h_2 dv + (\vec{\nabla} f)_w h_3 dw \quad -② \leftarrow a.b$$

$$\nabla f = (\vec{\nabla} f)_u \hat{i}_u + (\vec{\nabla} f)_v \hat{i}_v + (\vec{\nabla} f)_w \hat{i}_w \quad -ⓐ$$

$\left((\vec{\nabla} f)_r dr + (\vec{\nabla} f)_\theta r d\theta\right.$

①&② ⇒ comparing

$*(\vec{\nabla} f)_u = \frac{1}{h_1}\frac{\partial f}{\partial u}$ | $\frac{\partial f}{\partial u} = \vec{\nabla} f|_u h_1$

$(\nabla f)_v = \frac{1}{h_2}\frac{\partial f}{\partial v}$ | $\frac{\partial f}{\partial v} = \vec{\nabla} f|_v h_2$

$$\left(\vec{\nabla} f\right|_w = \frac{1}{h_3}\frac{\partial f}{\partial u} \qquad \frac{\partial f}{\partial w} = \vec{\nabla} f\Big|_w h_3$$

$$\vec{\nabla} f = \frac{1}{h_1}\frac{\partial f}{\partial u}\hat{i}_u + \frac{1}{h_2}\frac{\partial f}{\partial v}\hat{i}_v + \frac{1}{h_3}\frac{\partial f}{\partial w}\hat{i}_w$$

$$\vec{\nabla} = \frac{\hat{i}_u}{h_1}\frac{\partial}{\partial u} + \frac{\hat{i}_v}{h_2}\frac{\partial}{\partial v} + \frac{\hat{i}_w}{h_3}\frac{\partial}{\partial w} \qquad \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}\right)$$

Div

$$\vec{A} = A_1\hat{i}_u + A_2\hat{i}_v + A_3\hat{i}_w$$

$$\vec{\nabla}\cdot\vec{A} =$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(A_1 h_2 h_3) + \frac{\partial}{\partial v}(A_2 h_1 h_3) + \frac{\partial}{\partial w}(A_3 h_1 h_2)\right]$$

$$\left(\nabla\cdot\vec{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \right.$$

$$\left. = \frac{1}{r}\left[\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta}\right]\right)$$

Curl

$$\vec{\nabla}\times\vec{A} =$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

$$= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{i}_u & h_2\hat{i}_v & h_3\hat{i}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

Cylindrical Coordinate System

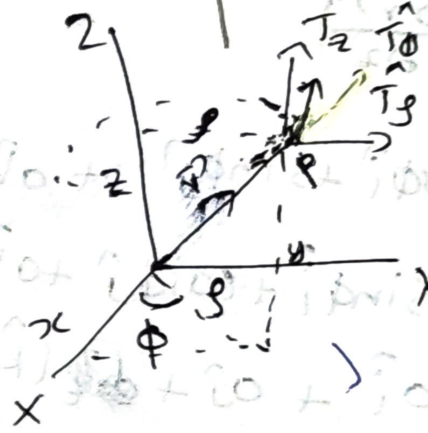
$\rho, \phi, z,$ $\quad u=\rho,\ v=\phi,\ w=z$

$u=x,\ v=y,\ w=z$

Volume

$\rho\, d\rho\, d\phi\, dz$

$\hat{i}_\rho,\ \hat{i}_\phi,\ \hat{i}_z$



$x = \rho\cos\phi$ $\quad \rho \geq 0$

$y = \rho\sin\phi$ $\quad 0 \leq \phi < \infty$

$z = z$ $\quad -\infty < z < \infty$

$\rho \to > 0$

$\phi \to 0$ to 2π

$z \to -\infty$ to ∞

$\rho^2 = x^2 + y^2$ $\qquad \tan\phi = \frac{y}{x}$

$\rho = \sqrt{x^2+y^2}$ $\qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$ $\qquad z = z$

$ds^2 = d\vec{r}\cdot d\vec{r} \Rightarrow dx^2 + dy^2 + dz^2$

$|ds^2|\ d\rho^2 + \rho^2 d\phi^2 + dz^2$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$

tangent to the curve.

$\hat{i}_\rho = \dfrac{\frac{\partial \vec{r}}{\partial \rho}}{\left|\frac{\partial \vec{r}}{\partial \rho}\right|}$ $\qquad \dfrac{\partial \vec{r}}{\partial \rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$\left|\dfrac{\partial \vec{r}}{\partial \rho}\right| = 1 = h_1$ (for linear curve $h_1 = 1$)

$\hat{i}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$\dfrac{\partial \vec{r}}{\partial \phi} = -\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}$

$\hat{i}_\phi = \dfrac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} =$ $\qquad \left|\dfrac{\partial \vec{r}}{\partial \phi}\right| = \sqrt{\rho^2(\sin^2\phi + \cos^2\phi)} = \rho$

$\vec{T}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$$\vec{T}_z = \frac{\frac{\partial \vec{r}}{\partial z}}{\left|\frac{\partial \vec{r}}{\partial z}\right|} = \hat{k}$$

$\hat{T}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} + 0\hat{k}$

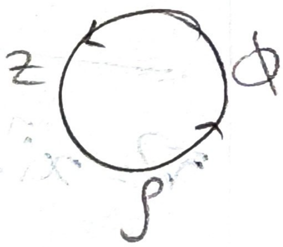
5 × 3 × 3 1

$\hat{T}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} + 0\hat{k}$

$\hat{T}_z = 0\hat{i} + 0\hat{j} + 1\hat{k}$

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix} = \begin{pmatrix} \hat{T}_\rho \\ \hat{T}_\phi \\ \hat{T}_z \end{pmatrix}$$

Calculate the inverse.



$T_\rho \cdot T_\phi = 0 \qquad T_\rho \times T_\phi = \hat{T}_z$

$T_\phi \cdot T_z = 0 \qquad T_\phi \times T_z = \hat{T}_\rho$

$T_z \cdot T_\rho = 0 \qquad T_z \times T_\rho = \hat{T}_\phi$

$$A^{-1} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since A is orthogonal

$A^T = A^{-1}$

$h_\rho = 1$ (linear) $\qquad h_\phi = \rho$

$h_z = 1$

Gradient

$\cancel{\vec{A} = A_\rho \hat{i}_\rho + A_\phi \hat{i}_\phi + A_z \hat{i}_z}$

f = scalar quantity

$$\vec{\nabla} = \frac{\hat{i}_u}{h_1}\frac{\partial}{\partial u} + \frac{\hat{i}_v}{h_2}\frac{\partial}{\partial v} + \frac{\hat{i}_z}{h_3}\frac{\partial}{\partial w}$$

Cylindrical system

$$\vec{\nabla} = \frac{\hat{i}_\rho}{1}\frac{\partial}{\partial \rho} + \frac{\hat{i}_\phi}{\rho}\frac{\partial}{\partial \rho} + \frac{\hat{i}_z}{1}\frac{\partial}{\partial z}$$

$$\vec{\nabla} f = \hat{i}_\rho \frac{\partial f}{\partial \rho} + \frac{\hat{i}_\phi}{\rho}\frac{\partial f}{\partial \rho} + \hat{i}_z \frac{\partial f}{\partial z}$$

$$\nabla^2 f = \vec{\nabla}\cdot\vec{\nabla} f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial w}\right)\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\left(\frac{\partial f}{\partial \rho} + \rho\frac{\partial^2 f}{\partial \rho} + \frac{1}{\rho}\frac{\partial^2 f}{\partial \phi^2} + \rho\frac{\partial^2 f}{\partial z^2}\right.$$

$$\cancel{= \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right)\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$A = (\rho, \phi, z)$, $\left.\begin{array}{l}\vec{\nabla}\cdot\vec{A} \\ \vec{\nabla}\times\vec{A}\end{array}\right\}$ HW

$$\nabla\cdot A \Rightarrow \frac{1}{h_\rho h_\phi h_z}\left[\frac{\partial(A_\rho h_\phi h_z)}{\partial\rho}+\frac{\partial(A_\phi h_\rho h_z)}{\partial\phi}+\frac{\partial(A_z h_\rho h_\phi)}{\partial z}\right]$$

$$\Rightarrow \frac{1}{\rho}\left[\frac{\partial(A_\rho \rho)}{\partial\rho}+\frac{\partial(A_\phi)}{\partial\phi}+\frac{\partial(A_z\rho)}{\partial z}\right]$$

$$=\frac{1}{\rho}\left[\rho\frac{\partial A_\rho}{\partial\rho}+A_\rho+\frac{\partial A_\phi}{\partial\phi}+\rho\frac{\partial A_z}{\partial z}\right]$$

$$=\frac{A_\rho}{\partial\rho}+\frac{1}{\rho}A_\rho+\frac{1}{\rho}\frac{A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}$$

$$\nabla\times A \Rightarrow \frac{1}{h_\rho h_\phi h_z}\begin{vmatrix} h_\rho\hat{I}_\rho & h_\phi\hat{I}_\phi & h_z\hat{I}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho h_\rho & A_\phi h_\phi & A_z h_z \end{vmatrix}$$

$$=\frac{1}{\rho}\begin{vmatrix} \hat{I}_\rho & \rho\hat{I}_\phi & \hat{I}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

example.

$\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ Express A is cylindrical coordinat system.

$$\vec{A} = A_\rho\hat{I}_\rho + A_\phi\hat{I}_\phi + A_z\hat{I}_z$$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

$$\begin{pmatrix}\hat{I}_\rho \\ \hat{I}_\phi \\ \hat{I}_z\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} i \\ j \\ k\end{pmatrix}$$

$\hat{T}_\rho = \frac{\partial \vec{r}}{\partial \rho} \Big/ \left|\frac{\partial \vec{r}}{\partial \vec{r}}\right| = \cos\phi\, \hat{i} + \sin\phi\, \hat{j}$

$\hat{T}_\phi = -\sin\phi\, \hat{i} + \cos\phi\, \hat{j}$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= \rho\cos\phi\, \hat{i} + \rho\sin\phi\, \hat{j} + z\hat{k}$

$\hat{T}_z = \hat{k}$

$A_\rho = \vec{A} \cdot \hat{T}_\rho$

$= \cancel{z\hat{i}\ 2x\hat{i}} +$

$= (z\hat{i} - 2\rho\cos\phi\, \hat{j} + \rho\sin\phi\, \hat{k}) \cdot (\cos\phi\, \hat{i} + \sin\phi\, \hat{j})$

$A_\rho = z\cos\phi - \rho\sin 2\phi$

$A_\phi = \vec{A} \cdot \hat{T}_\rho =$

$= -z\sin\phi - 2\rho\cos^2\phi$

$A_z = \vec{A} \cdot \hat{T}_z =$

$= \rho\sin\phi$

$\vec{A} = A_\rho \hat{T}_\rho + A_\phi \hat{T}_\phi + A_z \hat{T}_z$

$= (z\cos\phi - \rho\sin 2\phi)\hat{T}_\rho - (z\sin\phi + 2\rho\cos^2\phi)\hat{T}_\phi$

$+ \rho\sin\phi\, \hat{T}_z$

---

example - 2

$\vec{A} = (z\cos\phi - 2\rho\sin\phi\cos\phi)\hat{T}_\rho - (z\sin\phi + 2\rho\cos^2\phi)\hat{T}_\phi$

$+ \rho\sin\phi\, \hat{T}_z$

express A in cartesian

$$\begin{bmatrix} j \\ K \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_\phi \\ T_z \end{bmatrix}$$

Cordinate transformation matrix.

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z\cos\phi - 2\rho\sin\phi\cos\phi \\ -z\sin\phi - 2\rho\cos^2\phi \\ \rho\sin\phi \end{bmatrix}$$

$$= \begin{bmatrix} z\cos^2\phi - 2\rho\sin\phi\cos^2\phi + z\sin^2\phi + 2\rho\cos^2\phi\sin\phi \\ z\sin\phi\cos\phi - 2\rho\sin^2\phi\cos\phi - z\sin\phi\cos\phi - 2\rho\cos^3\phi \\ \rho\sin\phi \end{bmatrix}$$

$$= \underset{-2x}{z} = \begin{bmatrix} z \\ 2\rho\cos\phi(1) \\ y \end{bmatrix}$$

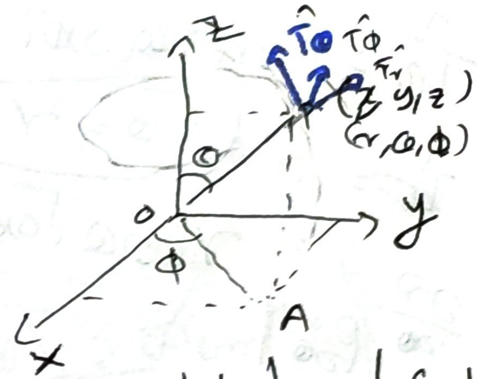
$A_x = z$

$A_y = -2x$

$A_z = y$

Spherical polar coordinate system

(r, θ, ϕ) $\quad r =$ intersection of $\theta = c$ & $\phi = d$



right hand system

relation between Cartesian and Spc

$OA = r\sin\theta$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

Surfaces are mutually $\perp^r$

$r^2 = x^2 + y^2 + z^2$ $\qquad r = \sqrt{x^2+y^2+z^2}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|ds|^2 = ds \cdot ds = dx^2 + dy^2 + dz^2$

$\sqrt{dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2}$

$\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$dv = dr\; r d\theta\; r\sin\theta\, d\phi$

$\tan\phi = \frac{y}{x} =$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{T}_r = \cancel{\frac{\partial \vec{r}}{\partial r}} \quad \frac{\frac{\partial \vec{r}}{\partial r}}{\underbrace{\left|\frac{\partial \vec{r}}{\partial r}\right|}_{h_r}}$

$= r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

$\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$=$ $\boxed{h_r = 1}$ $\sin^2\theta + \cos^2\theta = 1$ (linear)

$\boxed{\hat{T}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}$

$$\vec{T}_\theta = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$

$$h_\theta = r$$

$$\hat{T}_\theta = \frac{\partial\vec{r}}{\partial\theta} \Big/ \left|\frac{\partial\vec{r}}{\partial\theta}\right| = \boxed{\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} = \hat{T}_\theta}$$

$$\vec{T}_\phi = -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}$$

$$h_\phi = r\sin\theta$$

$$\frac{\frac{\partial\vec{r}}{\partial\phi}}{\left|\frac{\partial\vec{r}}{\partial\phi}\right|} = \boxed{-\sin\phi\,\hat{i} + \cos\phi\,\hat{j} = \hat{T}_\phi}$$

$\phi \to \theta \to r \to \phi$

$\vec{\nabla} f$, $\vec{\nabla}^2 f$, $\vec{\nabla}\cdot\vec{A}$, $\vec{\nabla}\times\vec{A}$

Example, $\vec{A} = x\hat{i} + 2y\hat{j} + yz\hat{k}$ in spc

Some use full formula in spc

$h_r = 1$ $\quad$ $h_\theta = r$ $\quad$ $h_\phi = r\sin\theta$

$\frac{\partial e_r}{\partial r} = 0$ $\quad$ $\frac{\partial e_r}{\partial\theta} = e_\theta$ $\quad$ $\frac{\partial e_r}{\partial\phi} = \sin\theta\, e_\phi$

$\frac{\partial e_\theta}{\partial r} = 0$ $\quad$ $\frac{\partial e_\theta}{\partial\theta} = -e_r$ $\quad$ $\frac{\partial e_\theta}{\partial\phi} = \cos\theta\, e_\phi$

$\frac{\partial e_\phi}{\partial r} = 0$ $\quad$ $\frac{\partial e_\phi}{\partial\theta} = 0$ $\quad$ $\frac{\partial e_\phi}{\partial\phi} = -\sin\theta\, e_r - \cos\theta\, e_\theta$

indipend on r

$$\cdot A = \frac{1}{h_r \ldots}\left[\ldots h_\theta h_\phi) + \frac{\partial}{\partial \theta}(A_\theta h_r h_\phi) + \frac{\partial}{\partial \phi}(A_\phi h_r h_\theta)\right]$$

$$= \frac{1}{r^2 \sin\theta}\left[\ldots \sin\theta) + \frac{\partial}{\partial \theta}(A_\theta r \sin\theta) + \frac{\partial}{\partial \phi}(A_\phi r)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(A_r r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \cdot \nabla f = \nabla^2 f = \frac{1}{h_r h_\theta h_\phi}\left(\frac{\partial}{\partial r}\left(\frac{h_\theta h_\phi}{h_r}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{h_r h_\phi}{h_\theta}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{h_r h_\theta}{h_\phi} \ldots\right.\right.$$

$$= \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}\left(\frac{r^2 \sin\theta}{1}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial \phi} \ldots\right.\right.$$

$$= \frac{1}{r^2 \sin\theta}\left(\frac{\partial f}{\partial r}\sin\theta\, 2r + r^2 \sin\theta \frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{\partial \theta}\cos\theta + \sin\theta \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin\theta}\frac{\partial f}{\partial \phi^2}\right.$$

$$= \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{\tan\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin\theta}\frac{\partial f}{\partial \phi^2}$$

$$\nabla \times \vec{A} = \frac{1}{h_r h_\theta h_\phi} \begin{bmatrix} \hat{i}_r h_r & \hat{i}_\theta h_\theta & \hat{i}_\phi h_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ h_r A_r & h_\theta A_\theta & h_\phi A_\phi \end{bmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \begin{bmatrix} \hat{i}_r & \hat{i}_\theta r & r\sin\theta\, \hat{i}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{bmatrix}$$

$\vec{A} = x\hat{i} + 2y\hat{j} + yz\hat{k}$ in Spc

$$\begin{bmatrix} i_r \\ i_\theta \\ i_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & \cancel{\cos\phi} -\sin\theta \\ \cancel{\cos\theta}\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix}$$

$$\vec{A} = \begin{bmatrix} x \\ 2y \\ yz \end{bmatrix} = \begin{bmatrix} r\sin\theta\cos\phi \\ 2r\sin\theta\sin\phi \\ r^2\sin\theta\sin\phi \end{bmatrix}$$

$\cancel{\sin}\ \vec{A} =$

$$\rightarrow \begin{bmatrix} r\sin^2\theta\cos^2\phi + 2r\sin^2\theta\sin^2\phi + r^2\sin\theta\sin\phi\cos^2\theta \\ r\sin\theta\cos\theta\cos^2\phi + 2r\sin\theta\cos\theta\sin^2\phi - r^2\sin^2\theta\sin\phi\cos\theta \\ -r\sin\theta\cos\phi\sin\phi + 2r\sin\theta\sin^2\phi\cos\phi + 0 \end{bmatrix}$$

$= r\sin\theta($

$$\hat{T}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{T}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{T}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} + 0\,\hat{k}$$

$$\begin{bmatrix} \hat{T}_r \\ \hat{T}_\theta \\ \hat{T}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \; \cancel{\cos\theta\sin\phi} & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix}$$

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{T}_\rho \\ \hat{T}_\phi \\ T_z \end{bmatrix}$$

~~T_r θ~~ ~~extent~~ In cylindrical

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{T}_r \\ \hat{T}_\theta \\ \hat{T}_\phi \end{bmatrix}$$

$\cos\theta$

$$\begin{bmatrix} T_r \\ T_\theta \\ T_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\theta & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{T}_\rho \\ \hat{T}_\phi \\ \hat{T}_z \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos^2\phi + \sin\theta\sin^2\phi & -\sin^2\theta\cos\theta + \sin\theta \\ & \end{bmatrix}$$

$$\vec{\nabla} f = \frac{1}{h_1}\frac{\partial f}{\partial r}\hat{i}_r + \frac{1}{h_2}\frac{\partial f}{\partial \theta}\hat{i}_\theta + \frac{1}{h_3}\frac{\partial f}{\partial \phi}\hat{i}_\phi$$

$$= \frac{\partial}{\partial r}\hat{i}_r + \frac{\partial}{\partial \theta}\frac{\hat{i}_\theta}{r} + \frac{\partial}{\partial \phi}\frac{\hat{i}_\phi}{r\sin\theta}$$

1) $f(r,\theta,\phi) = r^2\sin\phi$

$$\vec{\nabla} f = \sin\phi \, 2r\hat{i}_r + \cancel{\frac{r^2\cos\theta}{r}} + \frac{r^{\cancel{2}}\cos\phi}{\cancel{r}\sin\theta}$$

Tensor

Afken

Tensor is a General form any

1. Rank (n)
2. space (d Dimention)
3. It has componenents labeled by n indices with each index assigned values from one to d. and ~~totel~~ there for having a totel of d^n components.

eg: α_{11} , α_{123} , α_{1234}

The component transform under coordinate tranformation In a specific manner.

Rank(n)

components

$A_1 + \cancel{A_2 + A_3}$

$\alpha + \beta + \gamma$

To represent each component how many basis we ned.

for vetor 1 rank

$\vec{A}(\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k})$

rank 0 is a scalar.

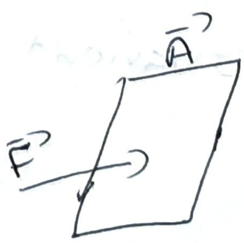
Rank 2 tensor

$A = \alpha_{11} + \beta_{12} + \gamma_{13}$

d^n = componets,
3^n = a compon...

stress

$\sigma = \frac{\vec{F}}{\vec{A}}$



$\sigma = \sigma_{11} + \sigma_{12} +$

$\sigma = \sigma(ii)$

$$\vec{\nabla} f = \frac{1}{h_1}\frac{\partial f}{\partial r}\hat{T}_r + \frac{1}{h_2}\frac{\partial f}{\partial \theta}\hat{T}_\theta + \frac{1}{h_3}\frac{\partial f}{\partial \phi}\hat{T}_\phi$$

$$\hat{T}_{u_j} = \frac{\partial \vec{r}}{\partial u_j} \Big/ \left|\frac{\partial \vec{r}}{\partial u_j}\right| \quad \text{tangent}, \ \hat{T}_{u_1}, \ldots \ \hat{T}_{u_2}$$

$r = xi + yj + zk$

$x =$
$y =$
$z =$

$$h_j = \left|\frac{\partial \vec{r}}{\partial u_j}\right|$$

$$d\vec{r} = \sum_j \frac{\partial \vec{r}}{\partial u_j} du_j$$

$$= \sum_j h_j \hat{T}_{u_j} du_j$$

$$d\vec{V} = \left|h_1 du_1 \hat{T}_{u_1} \cdot (h_2 du_2 \hat{T}_{u_2} \times h_3 du_3 \hat{T}_{u_3})\right| = \quad \left(\hat{T}_{u_1} \cdot (\hat{T}_{u_2} \times \hat{T}_{u_3})\right) = 1$$

$$= h_1 h_2 h_3 du_1 du_2 du_3 \quad (e_1 \cdot (e_2 \times e_3)) = 1 \quad \Rightarrow dv = J\, du_1 du_2 du_3$$

$$\frac{\partial \vec{r}}{\partial u_1} \cdot \frac{\partial \vec{r}}{\partial u_2} \times \frac{\partial \vec{r}}{\partial u_3} = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} \quad \left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right|$$

Jacobian determ

J of x, y, z w r to $u_1 u_2 u_3$

u_1 const surface

$$d\vec{A}_{23} = h_2 du_2 \hat{T}_{u_2} \times h_3 du_3 \hat{T}_{u_3} = \underline{h_2 h_3 du_2 du_3}$$

$$ds^2 = d\vec{r} \cdot d\vec{r} = \sum_j h_j^2 du_j^2 \Rightarrow ds = \sqrt{}$$

$$\vec{\nabla} = \sum_j \frac{1}{h_j}\frac{\partial}{\partial u_j}\hat{T}_{u_j}$$

Tensor

Afken

Usefull formulas (Polar)

$h_r = 1$, $h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| =$, $\vec{r} = x\hat{i} + y\hat{j}$, $x = r\cos\theta$
$y = r\sin\theta$

$h_\theta = r$, $h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| =$

$\frac{\partial \hat{e}_r}{\partial \theta} = e_\theta$, $e_r = \frac{\frac{\partial \vec{r}}{\partial r}}{h_r} = \sin\theta\hat{i} + \cos\theta\hat{j}$

$\rightarrow \cos\theta\hat{i} - \sin\theta\hat{j}$

$\frac{\partial e_\theta}{\partial \theta} = -e_r \rightarrow -\sin\theta\hat{i} - \cos\theta\hat{j}$

$\frac{\partial e_r}{\partial r} = 0 \quad 0$

$r = 1$

$\frac{\partial e_\theta}{\partial r} = 0 \quad 0$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

General formula

$$\nabla^2\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_1 h_3}{h_2}\frac{\partial \phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial w}\right)\right]$$

you can derive it

$\nabla \cdot \nabla\phi \rightarrow$

$d\vec{r} = h_u \hat{T}_u du + h_v \hat{T}_v dv + h_w \hat{T}_w dw$

$dV =$

$$\vec{\nabla} f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{T}_{u_1} + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{T}_{u_2} + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{T}_{u_3} = \sum_j \frac{1}{h_j}\frac{\partial f}{\partial u_j}\hat{T}_{u_j}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial}{\partial u_2}(A_2h_1h_3) + \cdots\right]$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{bmatrix} h_1\hat{T}_{u_1} & h_2\hat{T}_{u_2} & h_3\hat{T}_{u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{bmatrix}$$

$$\vec{\nabla}\cdot\vec{\nabla}\phi = \nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \cdots\right]$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ cylindrical

$\hat{T}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} + 0\hat{k}$, $h_\rho = 1$

$\hat{T}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} + 0\hat{k}$, $h_\phi = \rho$

$\hat{T}_z = 0\hat{i} + 0\hat{j} + 1\hat{k}$, $h_z = 1$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

SPC

$\hat{T}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$\hat{T}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$\hat{T}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$h_r = 1$

$h_\theta = r$

$h_\phi = r\sin\theta$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Tensor

Afken

Tensor is a General form any

① Rank (n)

② space (d Dimention)

③ It has components labeled by n indices with each index assigned values from one to d. and ~~totel~~ there for having a totel of d^n components.

eg: α_{11}, α_{123}, α_{1234}

The component transform in a specific manner under coordinate tranformation

---

Rank(n)

components

$A_1 + A_2 + A_3$

$\alpha + \beta + \gamma$

To represent each component how many basis we ned.

for vetor 1 rank $\vec{A}(\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k})$

rank 0 is a scalar.

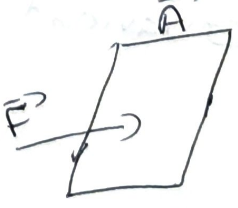
Rank 2 tensor

$A = \alpha_{11} + \beta_{12} + \gamma_{13}$

d^n = componets, 3^n = a compon...

Stress

$\vec{A}$



$\sigma = \frac{\vec{F}}{\vec{A}}$

$\sigma = \sigma_{11} + \sigma_{12} +$

$\sigma = \sigma(ij)$

$$\beta(x,y) = -\int_1^0 (1-\tau)^{x-1}\,\tau^{y-1}\,d\tau$$

$$= \int_0^1 \tau^{y-1}(1-\tau)^{x-1}\,d\tau$$

$$= \beta(y,x)$$

$\therefore$ β function is Symmetric

Trigonometric form of β function

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

if $x = \sin^2\theta$

$\Rightarrow$ $dx = 2\sin\theta\cos\theta\,d\theta$

limit

x	0	1
θ	0	$\pi/2$

$$\Rightarrow \beta(a,b) = \int_0^{\pi/2} (\sin^2\theta)^{a-1}(\cos^2\theta)^{b-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2a-2+1}(\cos\theta)^{2b-2+1}\,d\theta$$

$$\boxed{\beta(a,b) = 2\int_0^{\pi/2} (\sin\theta)^{2a-1}(\cos\theta)^{2b-1}\,d\theta}$$

eg: $\beta(1,1) = 2\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta$ | $\beta(2,2) = 2\int_0^{\pi/2} \sin^3\theta\cos^3\theta\,d\theta$

etc

Introduction to tensors

Some examples of Scalars & ~~tensors~~ vectors

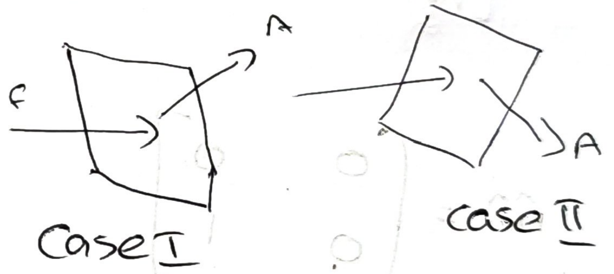
mass (scalar) , Speed (vector)

disstan (scalar , momentum (vector)

→ vector Contain more information than Scalars

→ vector are represented in Coordinates

sometimes the information needed to explain /denote/ represent a physical reality cannot be suffiently contained in a scalar or vector forms. A common exam is stress ~~p~~ inside a material. Consider following cases.

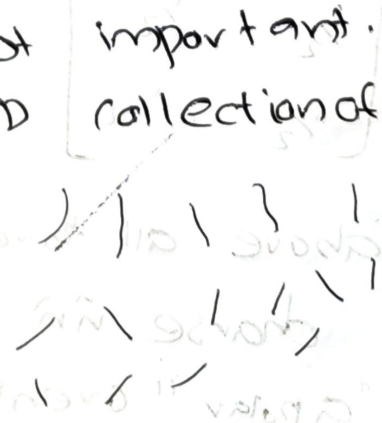


$\vec{F} = \underline{\text{Stress}}\ \vec{A}$

Can both the Cases can be explaied by single Scalar value of stress ?

Example - II : Headless field / orientation field

Let us consider unit vector $\hat{n}$. and Let us consider that we need to represent an 'a polar" field of unit vectors, meaning that the direction is not important. . you can imagine, for example a 2D collection of needle or ~~Road~~ rods .



and you wish to represent the local orientation.

1) If we stick a unit vector to each ~~st~~ needle/rod like this, $\hat{n}$ then obiously we ~~can~~ are differentiating $\hat{n}$ from , which means needles are polar like magnetic moments

2) If we take $\hat{n}\cdot\hat{n}$ it gives us 1 and and does not tell us abat orentation. If we take $\hat{n}\times\hat{n}$ it gives us 0. again no infor about orientation

3) what If we consider Something like $\hat{n} \otimes \hat{n}$

$$\hat{n}\hat{n}^T = \begin{bmatrix} n_x n_x & n_x n_y \\ n_y n_x & n_y n_y \end{bmatrix} ? \quad , \quad \hat{n} = \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

In this case:

$\Rightarrow$ $n_x = 0$, $n_y = 1$ or $n_x = 0$, $n_y = -1$ $\Rightarrow$ $\hat{n}\hat{n} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

180 rotated

$\Rightarrow$ $n_x = 1$, $n_y = 0$ or $n_x = -1$, $n_y = 0$ $\quad \hat{n}\hat{n} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

45 $\Rightarrow$ $n_x = \frac{1}{\sqrt{2}}$, $n_y = \frac{1}{\sqrt{2}}$ or $n_x = -\frac{1}{\sqrt{2}}$, $n_x = -\frac{1}{\sqrt{2}}$ $\quad \hat{n}\hat{n} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

NOW, we can rotate all ~~the~~ the above all three ~~needle~~ rods by 180°, It does not change $\hat{n}\hat{n}$ mean. we have constructed a "apolar" orentation field usin unit vectors $\hat{n}$ now it doesnot matter

If we stick this is unit vector to a rod like, $\hat{n}$ or
the field $\hat{n}\hat{n}^T$ is unchanged. here $\hat{n}\hat{n}^T$ is a tensor,

Example III

- Relation between magnetization (macroscopic) and applied magnetic field

$\vec{m} = \chi \vec{H}$ ← susceptibility

If χ is scalar, means the material magnetization $\vec{m}$ is ~~also~~ also in the direction of $\vec{H}$. But for some complex/exotic materials, the magnetization $\vec{m}$ may not be in the direction of $\vec{H}$. in such cases we cannot consider χ as a scalar (because it give the same direction) or a vector

In such cases $\vec{m} = \overline{\overline{\chi}}\vec{H}$, $\overline{\overline{\chi}}$ is a second order tensor

which gives
$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}$$

Example IV : Tensor as an outer product of two column vectors

If $\vec{E} = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$, the $\vec{E}\vec{E}^T = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \begin{bmatrix} E_x & E_y & E_z \end{bmatrix}$

$$\begin{bmatrix} E_xE_x & E_xE_y & E_xE_z \\ E_yE_x & E_yE_y & E_yE_z \\ E_zE_x & E_zE_y & E_zE_z \end{bmatrix}$$ second order

example $\underline{V}$; Gradient of velocity field as a vector tensor

ie $\vec{\nabla}\vec{v} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}$

not that $\vec{\nabla}\vec{v}$ is is diffent that $\vec{\nabla}\cdot\vec{v}$ or $\vec{\nabla}\times\vec{v}$

∇f - scalar

u chane, in space not get confuse

info about inflow or outflow

rotation

Example $\underline{VI}$: Electric current and conductivity

$\vec{J} = \overline{\overline{\sigma}}\vec{E}$ → Electrifield

→ conductivity tensor

charge flow/unit area/unit time

$J_1 = \sigma_{11}E_{11} + \sigma_{12}E_2 + \sigma_{13}E_3$

$J_2 = \sigma_{21}E_{11} + \sigma_{22}E_2 + \sigma_{23}E_3$

$J_3 = \sigma_{31}E_1 + \sigma_{32}E_2 + \sigma_{33}E_3$

Example $\underline{VII}$: Electric polerisation in an anisotropic medium

Let us suppose that we have an "anisotropic" medium which polerizes on application of an electric field $\vec{E}$

let us denot the polerization by vector $\vec{P}$

Becase the medium is anisotropic the polerization $\vec{P}$ need not be coliner with $\vec{E}$ (means need not be

In such a case

$$P_x = \gamma_{xx} E_x + \gamma_{xy} E_y + \gamma_{xz} E_z$$

$$P_y = \gamma_{yx} E_x + \gamma_{yy} E_y + \gamma_{yz} E_z$$

$$P_z = \gamma_{zx} E_x + \gamma_{zy} E_z + \gamma_{zz} E_z$$

or

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \underbrace{\begin{bmatrix} \gamma_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \gamma_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \gamma_{zz} \end{bmatrix}}_{\text{polarizability tensor of rank 2}} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

which in inde notatin can also write as

$$P_i = \sum_j \gamma_{ij} E_j$$ or in Eistein conventin $$P_i = \gamma_{ij} E_j$$

Extra Tensor is like a function, input vector and → output vector

Projection $\vec{q} = \bar{\bar{A}} \cdot \vec{p}$ $\vec{a}_1 = \begin{matrix} a_{11} \\ a_{12} \\ a_{13} \end{matrix}$

$\bar{\bar{A}} \cdot (\alpha \vec{m} + \beta \vec{n}) = \alpha \bar{\bar{A}} \cdot \vec{m} + \beta \bar{\bar{A}} \cdot n$

$\bar{\bar{A}} = \alpha_1 \vec{a}_1 \vec{b}_1 + \alpha_2 \vec{a}_2 \vec{b}_2 + \alpha_3 \underline{\vec{a}_3 \vec{b}_3} + \ldots$

Dyads. $\vec{a}\vec{b}$

Summation of dyads

componentof of a $\bar{\bar{A}}$ can be stored in 3x3 matrix

(3 basis $\{e_1, e_2, e_3\}$) → echvector components into

$\bar{\bar{A}} = \alpha_1 (a_{11} e_1 + a_{12} e_2 + a_{13} e_3)(b_{11} e_1 + \ldots b_{13} e_3) +$

$\alpha_2 (a_{21} e_1 + \ldots a_{23} e_3)(b_{21} e_1 + \ldots b_{23} e_3) + \ldots$

$\ldots$

$= A_{11} \vec{e}_1 \vec{e}_1 + A_{12} \vec{e}_{12} + A_{13} \vec{e}_1 \vec{e}_3 +$

$\ldots$

$A_{31} \vec{e}_3 \vec{e}_1 + \ldots + A_{33} \vec{e}_3 \vec{e}_3$

$$A = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = \vec{e}^T A \vec{e}$$

Null tensor → num matrix

$\bar{\bar{0}} \cdot \vec{p} = \vec{0}$

$$\underline{0} = \underset{\sim}{\vec{e}} \cdot 0 \cdot \underset{\sim}{\vec{e}}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

unit tensor → identity matrix

$\underline{I} \rightarrow I\vec{p} = \vec{p}$

$\underline{I} = \vec{e} \cdot I \cdot \vec{e}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow I = \vec{e_1}\vec{e_2} + \vec{e_2}\vec{e_2} + \vec{e_3}\vec{e} =$

conjugate tensor → transpose matrix

$\underline{A} = \vec{e} \cdot A \cdot \vec{e}^T \rightarrow \underline{A}^T = \vec{e} \cdot A^c \cdot \vec{e}^T$

$$\vec{\nabla} f = \frac{1}{h_1}\frac{\partial f}{\partial r}\hat{i}_r + \frac{1}{h_2}\frac{\partial f}{\partial \theta}\hat{i}_\theta + \frac{1}{h_3}\frac{\partial f}{\partial \phi}\hat{i}_\phi$$

$$= \frac{\partial}{\partial r}\hat{i}_r + \frac{\partial}{\partial \theta}\frac{\hat{i}_\theta}{r} + \frac{\partial}{\partial \phi}\frac{\hat{i}_\phi}{r\sin\theta}$$

1) $f(r,\theta,\phi) = r^2 \sin\phi$

$$\vec{\nabla} f = \sin\phi\, 2r\,\hat{i}_r + \cancel{\frac{r^2}{r}\cos\theta} \; \frac{r^{\cancel{2}}\cos\phi}{\cancel{r}\sin\theta}$$

Tensor

Afken

Tensor is a General form any

1. Rank (n)
2. Space (d Dimention)
3. It has componenents labeled by n indices with each index assigned values from one to d. and ~~totel~~ there fur having a totel of d^n components.

eg: α_{11}, α_{123}, α_{1234}

The component transform in a specific manner under coordinate tranformation

---

components

Rank(n)

$$\cancel{A_1 + A_2 + A_3}$$

$$\alpha + \beta + \gamma$$

To represent each component how may basis we ned.

for vetor 1 rank

$\vec{A}(\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k})$

rank 0 is a scalar.

Rank 2 tensor

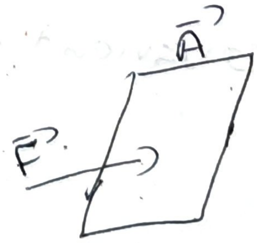
$$A = \alpha_{11} + \beta_{12} + \gamma_{13}$$

d^n = components,

3^n = a compon...

Stress

$$\sigma = \frac{\vec{F}}{\vec{A}}$$



$$\sigma = \sigma_{11} + \sigma_{12} +$$

$$\sigma = \sigma(ij)$$

wed. 12.10pm 10 marks MCQ , Matrix ... 2 P.1 / 8

Covarient , contravarient

Scalar $\longrightarrow$ Tensor rank zero

$\hookrightarrow$ Doesnot change under Cordinate transfor mati..

$$\phi(x^1, x^2, x^3), \quad \phi'(x'^1, x'^2, x'^3)$$

Gradient of ϕ $\quad x^i \to x'^i$

$d\phi = \nabla\phi \cdot dr$

$$\cancel{\frac{\partial\phi}{\partial x_i}} \quad \frac{\partial\phi}{\partial x'^i} = \frac{\partial\phi}{\partial x^1}\frac{\partial x^1}{\partial x'^i} + \frac{\partial\phi}{\partial x^2}\frac{\partial x^2}{\partial x'^i} + \frac{\partial\phi}{\partial x^3}\frac{\partial x^3}{\partial x'^i}$$

$$= \sum_j \frac{\partial x^j}{\partial x'^i}\frac{\partial\phi}{\partial x^j}$$

eg: $a_1x^1 + a_2x^2 + \dots + a_nx^n$

$$\sum_{j=1}^{n} a_i x^i$$

$$= a_i x^i$$

$$\frac{\partial\phi}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial\phi}{\partial x^j}$$

$$\frac{\partial\phi}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial\phi}{\partial x^j}$$

The components varying same transform fun as the chae of basis

$\hookrightarrow$ covarient factor

watch eugene tensor youtube

covarient tensor (~~I think its contra~~)

$$\rightarrow A'_i \sum_j \frac{\partial x^j}{\partial x'^i} A_j$$

$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j \rightarrow$$ Tranformation rule.

(dot product)

Contravarient tensor

$$dx'^i = \frac{\partial x'^i}{\partial x^1} dx^1 + \frac{\partial x'^i}{\partial x^2} dx^2 + \frac{\partial x'^i}{\partial x^3} dx^3$$

$$= \sum_j \frac{\partial x'^i}{\partial x^j} dx^j$$

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j$$

(displacement vector)

the components varrys with the inverse transfoomation to that of change of basis (in the) (normal vector rep)

Ingeneral

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j$$

Rank 2

~~Covarient~~ contravarient:

Inorder for this matter field to be a tensor it must obey following tranformation laws

$$(A')^{ij} = \sum_{kl} \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x'^{j}}{\partial x^{l}} A^{kl}$$ up running

mixed

$$(B')^{i}_{j} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{j}} B^{k}_{l}$$

contr

unpri

Covarient

$$C'_{ij} = \frac{\partial x^{k}}{\partial x'^{i}} \frac{\partial x^{l}}{\partial x'^{j}} C_{kl}$$

mixed

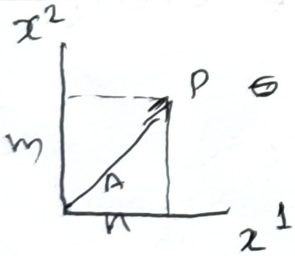
contr

$$X'^{ij}_{k} = \frac{\partial x'^{i}}{\partial x^{m}} \frac{\partial x'^{j}}{\partial x^{n}} \frac{\partial x^{p}}{\partial x'^{k}} X^{mn}_{p}$$ rank 3

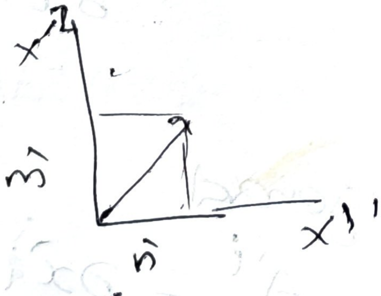
co

Contravarient

$A =$



$A = nx^1 + mx^2$



$x'^1 = \frac{x^1}{2}$, $x'^2 = \frac{x^2}{2}$ $A = n'x'^1 + m'x'^2$

$$A' = n'\frac{x^1}{2} + m'\frac{x^2}{2}$$

$$= nx^1 + mx^2$$

$\Rightarrow n' = 2n$

$m' = 2m$

Addition of two tensor

$\overline{\overline{A}}, \overline{\overline{B}}$ If they have Same rank, (covarient or co. or mixed)
in the same basis or space.

$A = A_{11} x^1 x^2 + A_{12} x^1 x^2 + \ldots A$

rank 1

$A_{vector} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$

(x^1, x^2, x^3)
basis

it need two bassis
~~it will have~~.

$$\overline{\overline{A}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ \vdots & \cdots & \vdots \\ A_{31} & & A_{33} \end{bmatrix} \text{ in } (x^1 \; x^2 \; x^3)$$

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ - & - & - \\ B_{31} & & B_{33} \end{bmatrix}$$

$(C_{ij} = A_{ij} + B_{ij})$

$C = A + B$

Symmetric tensor

$A^{ij} = A^{ji} \Rightarrow$ Symmetric matrix

$A^{ij} = -A^{ji} \rightarrow$ Antisymmetric

$$A^{ij} = \underbrace{\frac{1}{2}\left(A^{ij}+A^{ji}\right)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}\left(A^{ij}-A^{ji}\right)}_{\text{antisymmetric}}$$

$$A^{ij}+A^{ji} = A^{ji}+A^{ij} = A^{ij}+A^{ji}$$

$\llcorner$ swapping i & j

$A^{ji} - A^{ij} \longrightarrow$

$-\left(A^{ij}-A^{ji}\right)$

Product of two tensors

Direct product

A^{ij} and B_{kl} $\rightarrow$ Contra & covariant tensor.

$$C^{ij}_{kl} = A^{ij} B_{kl}$$

$$C'^{ij}_{kl} = A'^{ij} B'_{kl} = \underbrace{\frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q} A^{pq}}_{\text{contra}} \; \underbrace{\frac{\partial x^r}{\partial x'^k}\frac{\partial x^s}{\partial x'^l} B_{rs}}_{\text{Co}}$$

$$= \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\frac{\partial x^s}{\partial x'^l} A^{pq} B_{rs}$$

$\hookrightarrow$ Tensor of rank 4

Inner product

$i = k \rightarrow$ contraction

$$\bar{C}^{ij}_{il} = \underbrace{\frac{\partial x'^i}{\partial x^p} \frac{\partial x'^j}{\partial x^q}}_{\text{contra}} \underbrace{\frac{\partial x^p}{\partial x'^i} \frac{\partial x^s}{\partial x'^l}}_{\text{co}} C^{pq}_{ps}$$

$$= \underbrace{\frac{\partial x'^j}{\partial x^q} \frac{\partial x^s}{\partial x'^l} C^{pq}_{ps}}_{\text{rank - 2}}$$ mixed.

is similar to dot product.

Example

a_i and b^j two vector

$$c'^i_j = (\hat{a}_i)(b'^j)$$

$$= \frac{\partial x^k}{\partial x'^i} a_k \frac{\partial x'^j}{\partial x^l} b^l$$

$$= \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} \frac{a_k b^l}{\cancel{C^l_k}} \rightarrow C^l_k$$

$$c^i_j \quad \begin{matrix} l = j \\ k = l \end{matrix} \quad \frac{\partial x^l}{\partial x'^i} \frac{\partial x'^i}{\partial x^l} C^l_l$$

$$\underbrace{\longrightarrow}_{\text{Scalar}} = C^l_l$$

$B\lambda$ ...

$(B - \lambda I)x = 0$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow$$

$(1+\lambda) - \lambda(\lambda^2 - 1) = 0$

$(1+\lambda) - \lambda(\lambda+1)(\lambda-1) = 0$

$(\lambda+1)(1 - \lambda(\lambda-1))$

$\cancel{(\lambda+1)(1+\lambda(1-\lambda))} = 0$

$(\lambda+1)(1 - \lambda^2 - \lambda)$

$1 + \lambda - \lambda^3 + \lambda$

$-\lambda^3 + 2\lambda + 1 = 0$

$\lambda^3 - 2\lambda - 1 = 0$

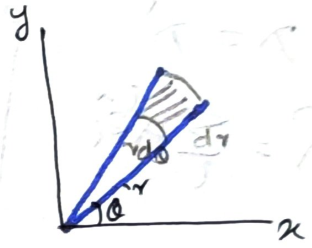
[illegible] 1

$x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\theta = \tan^{-1}(y/x)$

<u>Note</u>

bear in mind which quadrant the point lies

Slope in polar, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$

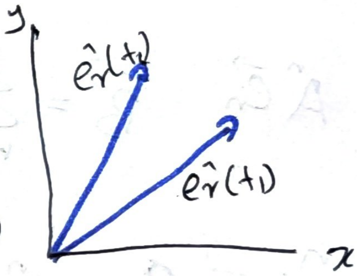


Area in polar co-ordinates,

$$\int_0^{\infty}\int_0^{2\pi} r\,dr\,d\theta$$

Vectors in plane polar coordinates

$\vec{r}(t) = r(t)\hat{e}_r$



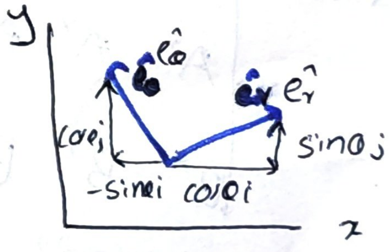
$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

(direction of $\hat{e}_r$ changes as t change)

(if θ changes with time)

so $\hat{e}_r = \hat{e}_r(\theta)$ $\Rightarrow \dfrac{d\hat{e}_r}{dt} = \dfrac{d\theta}{dt}\dfrac{d\hat{e}_r}{d\theta}$

$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$ (in cartesian)



$$\frac{d\hat{e}_r}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{e}_\theta$$

$\rightarrow \perp^r$ to $\hat{e}_r$ so $\dfrac{d\hat{e}_r}{d\theta} = \hat{e}_\theta$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e} + r\frac{d\theta}{dt}\hat{e}_\theta$$

$\hat{e}_r \cdot \hat{e}_\theta = 0$

basis.

any vector $\vec{u} = u_r\hat{e}_r + u_\theta\hat{e}_\theta$,

$$\vec{u} = (\vec{u}\cdot\hat{e}_r)\hat{e}_r + (\vec{u}\cdot\hat{e}_\theta)\hat{e}_\theta$$

$$\frac{de_\theta}{dt} = -e_r \,($$

Tensors

scalars invariant under cordinat transformation

$$\lambda = \lambda'$$

unit verovr

$\vec{A} = \sum_i A^i \hat{e}_i$ | $A^i = \vec{A} \cdot \hat{e}_i$ | Basis vector which are notunitue $\vec{e}_n \cdot \vec{e}_m = g_{nm}$

~~dirac de~~

Croneleor delta in 3D $\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

notunitvectr

$$\vec{A} = \sum_{n=1}^{3} A^n \vec{e}_n \qquad \vec{B} = \sum_{m=1}^{3} B^m \vec{e}_m$$

$$\vec{A} \cdot \vec{B} = \sum_{n=1}^{n} \sum_{m=1}^{n} g_{nm} A^n B^m$$

Cyclic permutation

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A^x & A^y & A^z \\ B^x & B^y & B^z \end{vmatrix}$$

Levi-Civita notat

$\varepsilon^{ijk} =$ 1 : 123, 231, 312,

-1 : 321, 213, 132

0 : 112 . . .

(General)

$$(\vec{A} \times \vec{B})^i = \sum_{k=1}^{3} \sum_{j=1}^{3} \varepsilon^{ijk} A^j B^k$$

~~represeted as old cordinated.~~

rotatin

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

$$z' = z$$

$$x' = x\frac{\partial x'}{\partial x} + y\frac{\partial x'}{\partial y}$$

$$y' = x\frac{\partial y'}{\partial x} + y\frac{\partial y'}{\partial y}$$

$$dx^{i'} = \sum_j x^{j} \frac{\partial x^{i'}}{\partial x^j}$$

~~$A^{x'} \sum A^x$~~ Defining a vector by component tranformation

$$\cancel{A^{x'} = A^x \cos\theta + A^y \sin\theta}$$

Vector

$$A^{i'} = \sum_i A^j \frac{\partial x^{i'}}{\partial x^j}$$

we still define θ as indisplacement pa

A^x is ~~part~~ component of vector ~~iff it tranform~~ same as the corresponding cordinate displacement. In orthogonal tranf..

$$A^{x'} = A^x \cos\theta + A^y \sin\theta$$

$$A^{y'} = A^x - \sin\theta + A^y \cos\theta$$

$$\cancel{A^z}\ A^{z'} = A^z$$

? $v^x = \frac{\partial x}{\partial t}$, vector component that are proportional to numerator displacements

$\nabla_x U = \frac{\partial U}{\partial x}$, $\frac{1}{\alpha}$ de nometor dispacemen

$$\begin{array}{cc} A & B \\ \begin{bmatrix} \\ \\ \end{bmatrix} & \begin{vmatrix} \\ \\ \end{vmatrix} \end{array}$$

~~$\sin\theta$ ... $-\sin\theta\cos$~~

$M^{ij} = \langle i|M|j\rangle$, $M = |A\rangle\langle B|$

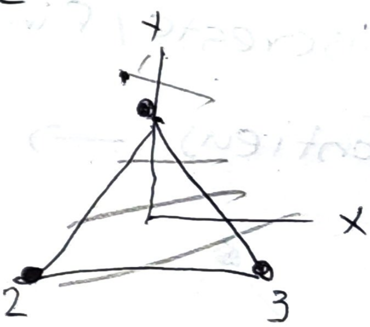
$$\sum_i a$$

Symmetry and Group theory (Arfken, weber)

A group G is defind as a set of objects or operations (eg: rotation or other transformation), called elements of G) that may be combimed by a procedure to be called mutiplicatin and denoted by $*$ to form a well defind product, subject to the following 4 conditin satisfy

1) If a and b are element of G
$a*b \Rightarrow$ element of G ((losure under mutipliacation)

2) mutiplication is associative
$a*(b*c) = (a*b)*c$



$C_3 + 12$

$C_3^2 \rightarrow 2h_0$ C_1, C_2'

3) There is a unique identity element in group G
$I*a = a*I = a$ for every element in G

4) each element 'a' in G It has a ~~ele~~ Inverse.
denote by a^{-1}

$aa^{-1} = I$

$a_1, a_2, a_2, a_4 \to$ this will form a group

Symmetry operation from groups

Discreate / fine groups (finite elements of the group)

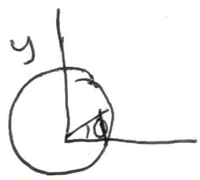
$D_3 \Rightarrow C_3, C_3^2, C_2, C_2', C_2'$ (Triangle)

Contieus group (infine groups and describe by continues parameter) circle

eg:  $R(\theta)$ $\theta \sim 2\pi$

1) Discreate / finite - elements

2) Contieus $\to$ ∞ elements.

eg:  

disk

$0 < \phi < 2\pi$

both are element of group.

$a * b = c$
$b * a = c'$ } $\in G$

$c \neq c'$ not commutative (non abelient group)

$c = c' \Rightarrow$ commutative (abelian group)

Group =

$I, a, a^2 (= aa), a^3, \ldots \Rightarrow$ Cyclic group.

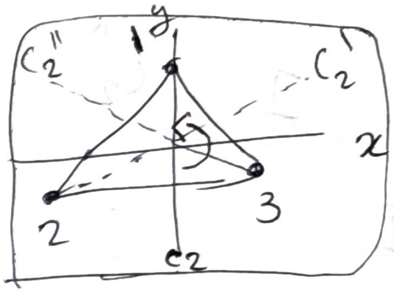
$a * a = a * a \Rightarrow$ abelian $\Rightarrow$ abelian & cyclic

But Not all abelian groups are cyclic

Finite group

Rotation usially anti clockwise.

$D_3 \Rightarrow$ Symmetry of equilateral triangle. (They are identical)



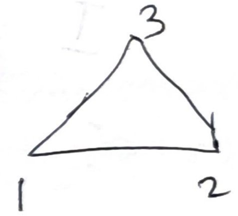
$-I$

* Cluster

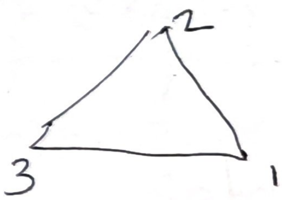
$C_3 \to 120^\circ$ rotation

$C_3^2 \to 240^\circ$

$\Rightarrow C_3 \times C_3 = C_3^2$



$- C_3$

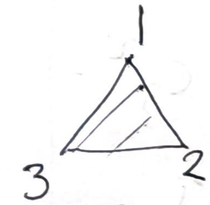


$- C_3^2$

$C_n =$ n fold Symmetry

$n = \frac{2\pi}{\theta}$ n fold axis

$C_2 \Rightarrow$ fliping along C_2 axis

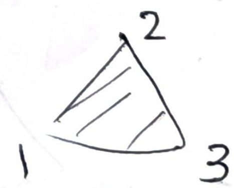


(mirror)

$C_2' \Rightarrow$ fliping C_2' axis

Sus
Fase Donot chane

$C_2'' \to$ flipig along C_2''



Mutiplication table

	I	C_3	C_3^2	C_2	C_2'	C_2''
C_3						
C_3^2						

Mutipliation table (D_3) Dihedral group

I	C_3	C_3^2	C_2	C_2'	C_2''
C_3	C_3^2	I	C_2''	C_2	C_2'
C_3^2	I	C_3	C_2'	C_2''	C_2
C_2	C_2'	C_2''	I	C_3	C_3^2
C_2'	C_2''	C_2	C_3^2	I	C_3
C_2''	C_2	C_2'	C_3	C_3^2	I

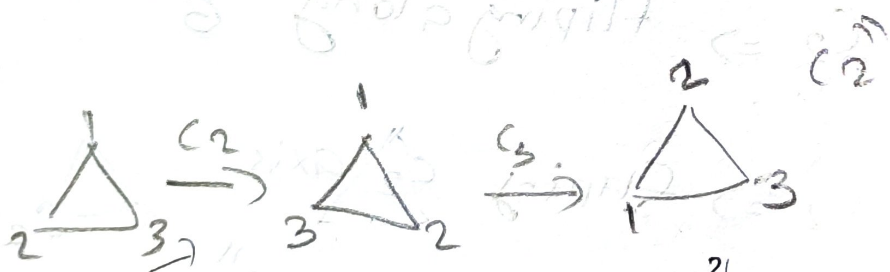
$\underline{I \times A = A}$

(Isomorphic same table)

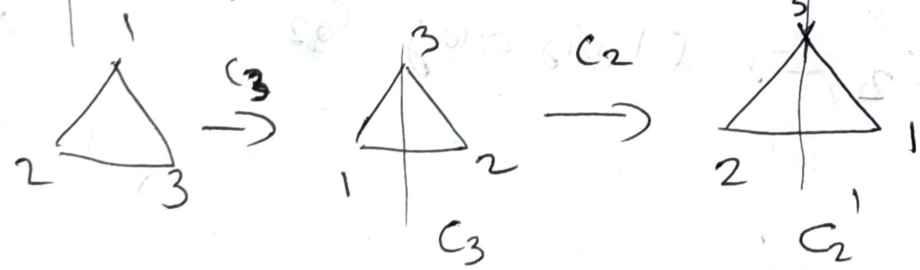
$C_3 \times C_3 = C_3^2$

$C_3 C_3^2 = I$

$C_3 C_2 =$



$C_2 C_3 =$



$\Rightarrow C_3 C_2 \neq C_2 C_3 \Rightarrow$ Non-Abelian group.

$aa^{-1} = I \Rightarrow C_3 C_3^2 = I \Rightarrow C_3 = {C_3^2}^{-1}$

$\Rightarrow C_3^2 C_3 \Rightarrow I \Rightarrow$

Symmetry operation & symbols, Symmetry elements

=> Geometrical fig., crystal, molecule,

I) Proper axis of symmetry => (Rotational Axis)

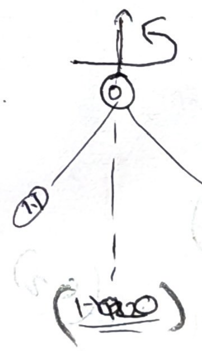
II) Plane of symmetry => (Reflection/ mirror symmetry)

III) Centre of symmetry => (Inversion of ~~centre of symmetry~~ centre)

IIII) Rotro reflection axis of symmety => (Improper axis)

I) Proper axis of symmety (D_3)

eg.:



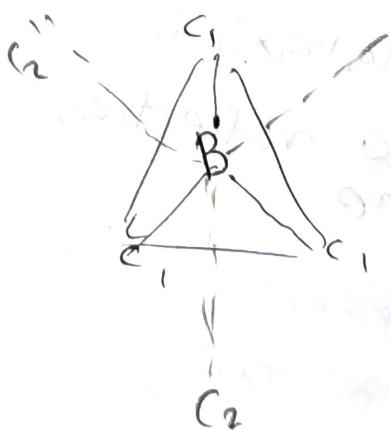
(180 rotation) π

=> Same.

$$\boxed{C_n = \frac{2\pi}{n} \Rightarrow n \text{ fold rotation}}$$

$$C_2 = \frac{2\pi}{2} = \pi$$

$$C_n^P = \frac{2\pi}{n} \times P$$



BCl_3

three 2 fold symmetry.

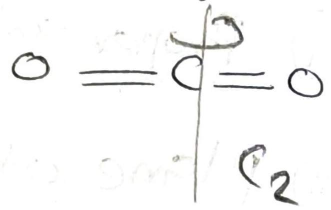
(C_2 C_2' C_2'')

N_2, ~~CO~~, CO_2 (~~linear~~)

Principle axis Homonuclear

Hetro nuclea NoSym...

→ Hetro Symmetric



~~Principle axis~~

(2) CH_4 @ How may Principal axis

4 C_3 & 2 C_2

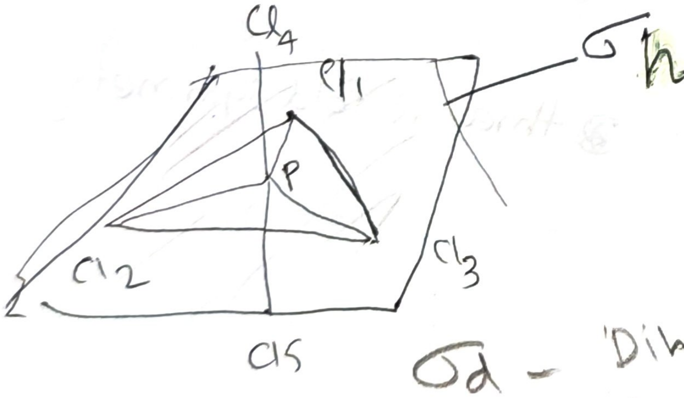
2) Symmetry plane | plane of symmetry

σ_v Principal axis palalell to (C_2)

mirror plane

plane contain Principal axis (Primary axis)

Sigma



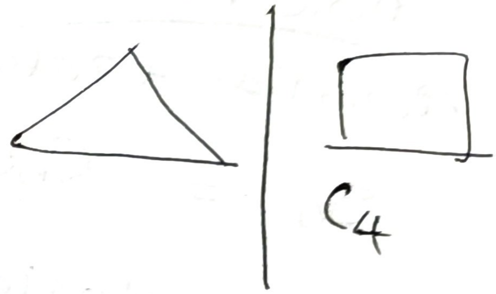
σ_h ⇒ Princlal axis is $\perp^r$ to reflection plane.

σ_d - Dihedrial plane.

Contain principal axis & Bisect the angle betwead C_2

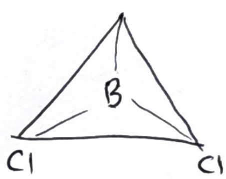
Principal axis

Highest symmetry axis (n) ⇒ eg: C_3 C_4



$\sigma_v^2 = I$

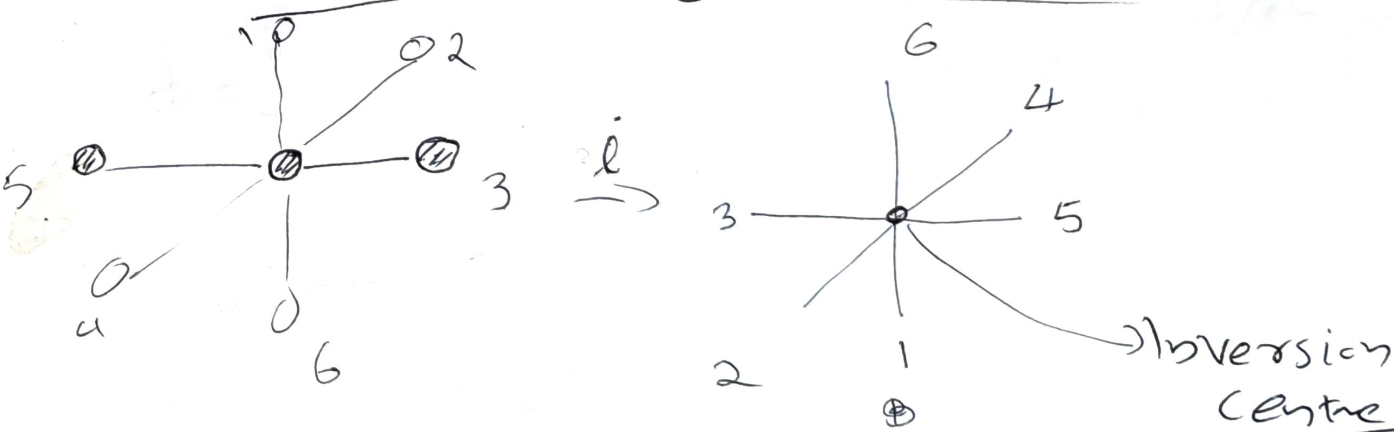
$\sigma_h^2 = I$



C_3 Symmetry

$\frac{360}{3} = 120$

Centre of Symmetry or Inversion Symmetry



$i^2 = I$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Roto reflection (element of symmetry
improper axis of symmetry)

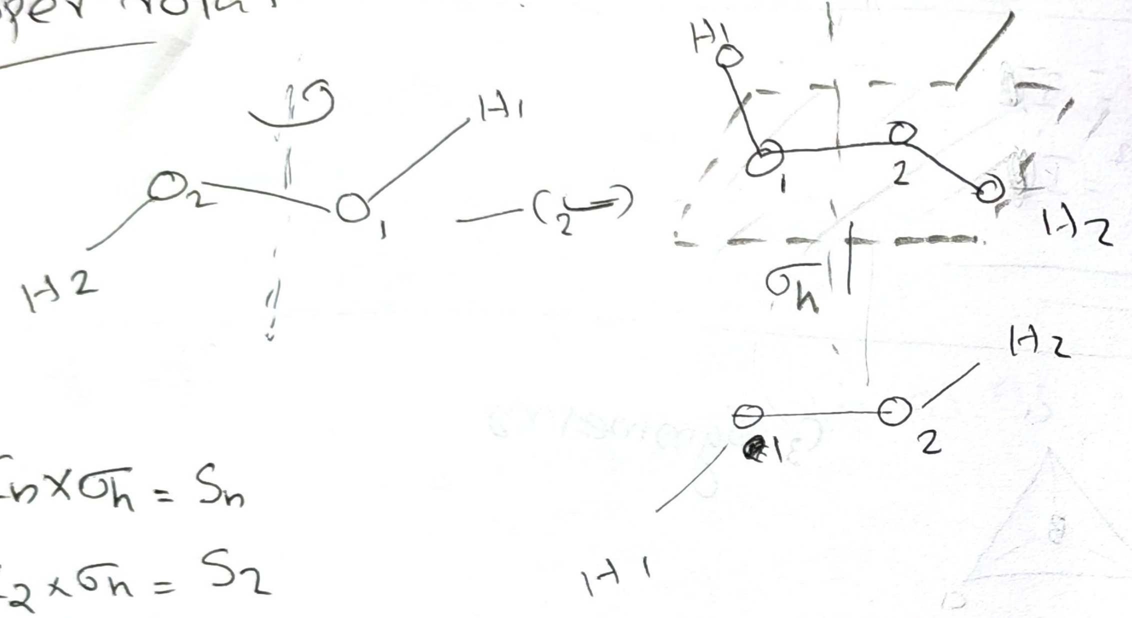
rotation + reflection

$S_n = C_n\sigma_h$ or σC_n

$\perp^r$ to axis of rotation

Improper rotation

H_2O_2



$C_n \times \sigma_h = S_n$

here $C_2 \times \sigma_h = S_2$

C_1 = only I

C_i = only inversion

C_s = only plane

$S_n \Rightarrow C_n\sigma$

$S_n^2 \Rightarrow C_n\sigma\, C_n\sigma = C_n^2\sigma^2 = C_n^2 I$

$S_n^3 \Rightarrow C_n\sigma\, C_n\sigma\, C_n\sigma = C_n^3\sigma^3 = C_n^3\sigma I$

⋮

$S_n^n \Rightarrow C_n^n\sigma^n = I \cdot I = \underline{\underline{I}}$

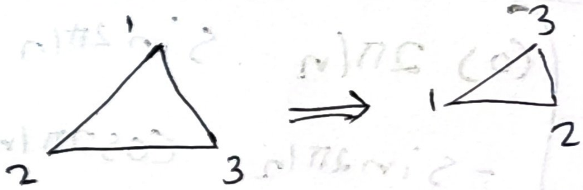
Point groups

molecular Symmetry group

Representation of groups

$C_3 \rightarrow 120^\circ$ rotation ~~in~~ In matrix

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$



$\theta = 120$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow C_3 = \begin{bmatrix} \cos 120 & \sin 120 \\ -\sin 120 & \cos 120 \end{bmatrix}$$

$$U(C_3) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\theta = 240$

$$U(C_3^2) = \begin{bmatrix} \cos 240 & \sin 240 \\ -\sin 240 & \cos 240 \end{bmatrix}$$

$$U(C_3^2) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$U(C_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(C_2 rotation about x axis ⇒ x unchged, y & z ⇒ y & −z)

$\overset{3D}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ about Y axis $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$U(C_2') = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$\left.\begin{aligned} C_2' &= C_3^2 C_2 \\ C_2'' &= C_3 C_2 \end{aligned}\right\} \text{check}$$

$$U(C_2'') = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

~~In 3D~~ In 3D rotation about z axis → reducable reprectation

$$\boxed{U(C_n) = \begin{bmatrix} \cos 2\pi/n & \sin 2\pi/n & 0 \\ -\sin 2\pi/n & \cos 2\pi/n & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$U(C_3^2) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

$\left.\begin{matrix} 2\times 2 \\ + \\ 1\times 1 \end{matrix}\right\}$ Reducable reprentation

can be redeced to irrreducable reprsen

⇒ $U(I) = 1$ (~~canot be broken to~~

⇒ $U(C_3) = U(C_3^2) = 1$

⇒ $U(C_2) = U(C_2') = U(C_2'') = -1$ ⇒ trival

Irreducable rep.

cannot be reduced into a direct sum of ~~smaller~~ representation of small dimension by application of some transformation

eg: $U(C_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $V = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -2/\sqrt{3} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \end{bmatrix}$

unitary tranformation

$V U(C_3) V^{-1} = ?$

Generator

$\{i, -1, -i, 1\} \Rightarrow i$ ~~there~~ $\quad i^2 = -1,\ i,\ i^3 = -i,\ i^4 = 1$

one element that can generate all other elemet

example- $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$Q = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$S = \begin{bmatrix} -1/2 & 1/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check D is generator,

this form ... (...) ... num ... de ... tion)

=> ... sat ... pertig

① ... real => ...

$$\begin{bmatrix} & 0 & \\ 0 & & \\ 0 & & \end{bmatrix} \begin{bmatrix} 0 & 0 & a \\ 0 & 1 & \\ 0 & 0 & \end{bmatrix} \begin{bmatrix} & & a_1+a_2 \\ & & b_1+b_2 \\ 0 & 0 & 1 \end{bmatrix}$$

② assosiative => check

3) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

4) inverse. $\begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$

How symmetry operation will form a groups -> they form Point group.

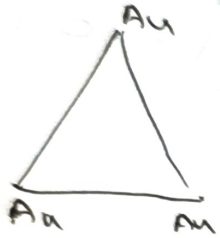
Point group => Molecular symmetry group.

=> symmetry operation -> (C_n, ..., σ, S)

molecule obey formal conditions of a group.

then the symmetry operation form a molecular gr

a point group.

eg: 

$D_3 \rightarrow$ Point group.

C_3 & C_2' Symmetry elements. ~~Point~~ Group.

4 types of point group (classification)

1) Non axial $\rightarrow$ No Rotational axis (except C_1)
only reflection (σ) or inversion (i)

$\Rightarrow C_1$ (only I), C_i (only i), C_s (only E, σ_h)

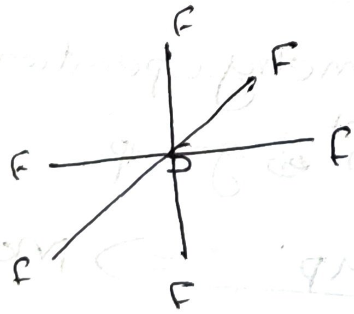
2) Axial $\Rightarrow$ It have n fold rotational symmetry

$\Rightarrow$ Improper axis (S_n)

3) Cubic point group.

It have several higher fold symmetry accompanied by some lower fold axis.

T, Td, Th, Oh,



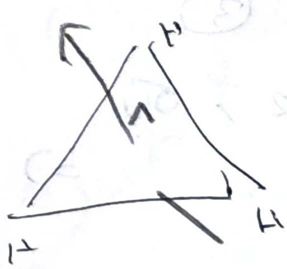
T - Tetrahedral
O - octahedral

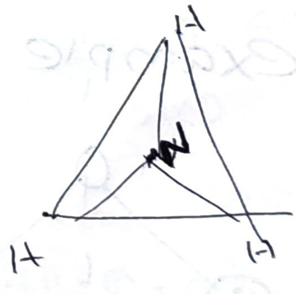
4) Special group → (linear molecule)

→ ∞ rotation

C_∞, $C_{\infty v}$, D_∞, $D_{\infty h}$ } infinet group.

Point group in molecule	and crystal
Symmetry elements	
C_n	n
D_n	n2
σ_v	m
σ_h	1m
C_{3v}	3m
C_{6h}	6/m
D_{3d}	3m2





D_3

C_{3v}

* How to find point group symmetry of molecules

order

① Non axial, cubic, special symmetry

(C_i, C_1, C_s) more than one high symmetry linear molecule

② If ① is not present then Axial symmetry

C_n, ($C_n + C_2'$) (rotation) or S_{nh} ⇒ C_n, D_n, S_{2n}

Point group

③ look for σ_h or σ_v (reflection) } C_{nh}, C_{nv}, D_{nh} } name

~~④ No look for σ~~

④ Then C_n or S_{2n} only

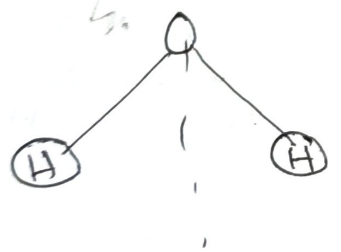
⇒ If C_n is also improper rotation axis

S_{2n} ✓ not C_n

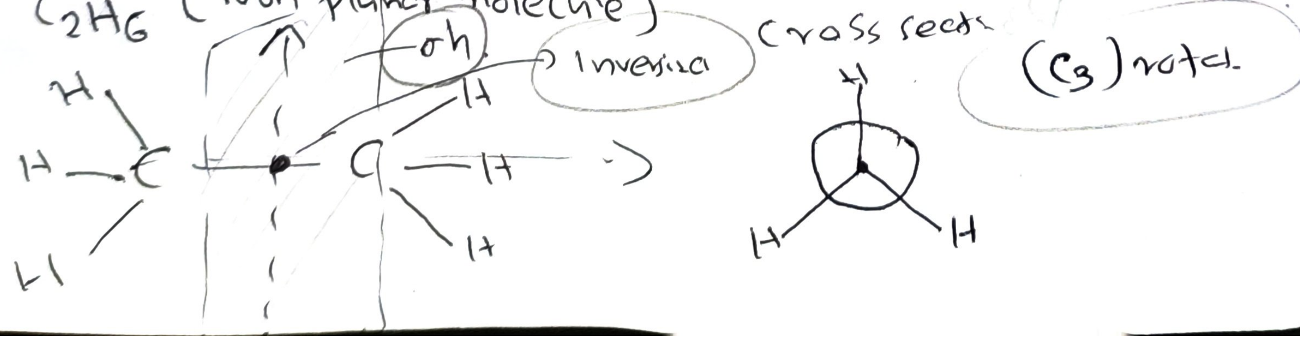
example for ③

$C_2 + \sigma_v$

⇒ C_{2v}



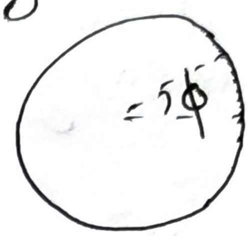
C_2H_6 (non planar molecule)



This case $C_{3h} \rightarrow C_3 + \sigma_h + C_i$

Contineous group :-

rotating Disc



$R(\phi) \rightarrow 0$ to 2π

group element varry from 0 to π

Ist element   second element

$$R(\phi) \times R(\theta) = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(1)

$$= \begin{bmatrix} \cos(\phi+\theta) & \sin(\phi+\theta) \\ -\sin(\phi+\theta) & \cos(\phi+\theta) \end{bmatrix}$$

(4) (Assosiatie)

group

$$U\phi = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

(2) identity $R(0)$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{pmatrix} \cos 0 & \sin 0 \\ -\sin 0 & \cos 0 \end{pmatrix}$

(3) Inverse. $R(2\pi - \theta)$ [If we multiple 0 or 2π

$R(-\theta)$ $0 - \theta$ $(2\pi - \theta + a)$

$= 2\theta$

~~check~~

Abelian or not (commutative)

$$R(\theta + \phi) = R(\phi + \theta) \Rightarrow \text{Abelian}$$

Lie group (rotation) (contieuus)

rotation
groups having Infinite number of elements

SO(n) -> why it is special because it have deter-
minant = 1 & angle varries continously
$\underbrace{\hspace{6em}}_{\text{Sus}}$

- Rotation is

This group is an exaple of $\underline{\underline{SO(2)}}$
special orthogonal in 2 Dimention

$$U(\phi) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$\boxed{\text{in general } SO(n)}$

$\boxed{SU(n)}$ special unitary.

---

rotation + reflection

O(n)

Generator → Generate all other element.

In case of Lie group generator is more important

Let's say Generator is S

The representation of element correspondane to Infinitisimal rotation $\delta\phi$

$$U(\delta\phi) = I + \delta\phi\underbrace{(iS)}_{\text{Generator}}$$

For large Values of ϕ

$$U(\phi) = \lim_{N\to\infty}\left(1+\frac{i\phi}{N}S\right), \qquad \delta\phi = \frac{\phi}{N}$$

Binomial series expansion

$$U(\phi) = e^{i\phi S}$$

Properties of S

For groups $SO(n)$ and $SU(n)$ and U will be Unitary → because orthogal is a special case of unitary.

$$U^{-1} = U^{\dagger}$$

$$U^{-1} = e^{(-i\phi S)}$$

$$U^{\dagger} = e^{-i\phi S^{\dagger}}$$

$$= U^{\dagger} = e^{(-i\phi S^{\dagger})}$$

$\Rightarrow S = S^{\dagger} \Rightarrow S$ is Hermitian

ii) $\det(U) = 1$, $\det(U) = \exp[i\phi \ \mathrm{trace}(S)]$

$= 1$

$\Rightarrow$ trace $= 0$ for any real non zero value of ϕ

$\boxed{\text{S is hermition and trace zero}}$

SO(2) Generator

For an infinitisimal rotation $\delta\phi$

$$(x', y') = (x + y\delta\phi , y - x\delta\phi)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \delta\phi \\ -\delta\phi & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \underline{I + i\delta\phi S}$$

$$iS = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \cdot \sigma_2$$

~~Paulises~~ pauli's matrix.

SU(2) Generator

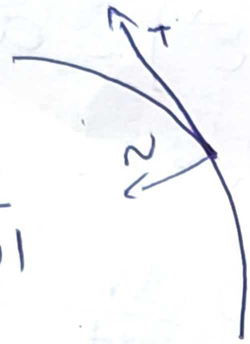
$S_i = \frac{1}{2}\sigma_j$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

They are Hermitian and trace 1.

Rotation denoted as α_j in connection with generator S_j, ~~when~~ we have remember SU(2)
as, $U_j(\alpha_j) = e^{i\alpha_j\sigma_j/2}$

$\vec{r} = \vec{r}(t)$

$\vec{r}'(t) =$ slope



unit target vector $= T = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} = \dfrac{v(t)}{|v(t)|}$

unit normal vector $N = \dfrac{v'(t)}{|v'(t)|}$

---

tangential component of acceleration =

$$a_T = a \cdot \hat{T} = \frac{\vec{v}(t) \cdot \vec{a}}{|\vec{v}(t)|}$$

normal

$$a_N = a \cdot \hat{N} = \frac{\vec{v}(t) \times \vec{a}}{|\vec{v}(t)|}$$

$\nabla^2 \phi = \nabla \cdot \nabla \phi$ | $\nabla \times (\nabla \phi) = 0$ & $\nabla \cdot (\nabla \times \vec{B}) = 0$

$|AB| = |A||B|$

$|cA| = c^n |A|$

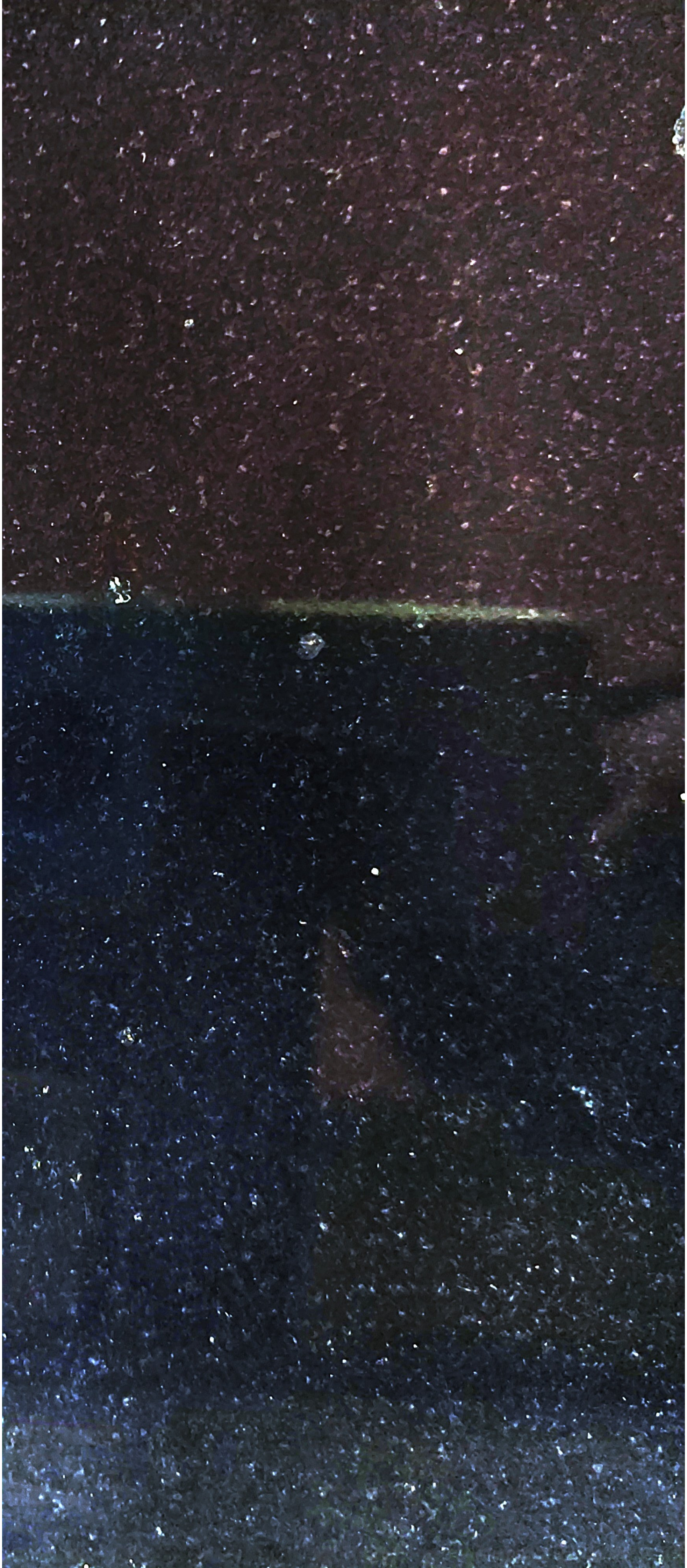
A & A^T has same eigen vales

$|A| = 0 \Rightarrow$ cloumn vectors are lineny dependend.

Trace = Sum of elements of main diagonal

= sum of eigen valu

= invarient under chage of basi



C_n = n fold Symmetry ($\frac{2\pi}{n} = {}^\circ$ rotation / $C_n^p = \frac{2\pi \times p}{n}$

Proper axis → C_3, C_2

Plane → σ_v, σ_h

Principl axis llel to σ ⇒ σ_v

⊥r ⇒ σ_h

centre → i

Roto reflectio → $C_n \sigma_h = S_n$

$$U C_n = \begin{bmatrix} \cos 2\pi/n & \sin 2\pi/n & 0 \\ -\sin 2\pi/n & \cos 2\pi/n & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

but,

$$U C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ about x

molecular point group

nonaxial → No rotation

C_1, C_i, C_s,

Axial → n fold rotation

Cubic → higher order T, Td, Th, Oh

special group → C_∞, $C_{\infty v}$, D_∞, $D_{\infty h}$

Contineas. group

Lie group ⇒ Rotational group having ∞ elements.

Tensor

$\hat{n} \otimes \hat{n} \Rightarrow \hat{n}\hat{n}^T$ eg: $\vec{\nabla}\vec{v} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} v_x & v_y & v_z \end{bmatrix} =$

D^n components, n = rank

$P_i = \sum_j \gamma_{ij} E_j = \gamma_{ij} E_j$

$x' = x\cos\theta + y\sin\theta \quad = \quad x' = x\frac{\partial x'}{\partial x} + y\frac{\partial x'}{\partial y} =$

$y' = -x\sin\theta + y\cos\theta \qquad y' = x\frac{\partial y'}{\partial x} + y\frac{\partial y'}{\partial y} =$

$z' = z$

$\Rightarrow \quad x'^i = \sum_j x^j \frac{\partial x'^i}{\partial x^j} \Rightarrow \quad x^j \frac{\partial x'^i}{\partial x^j} = x'^i$

in general $= A'^i = A^j \frac{\partial x'^i}{\partial x^j}$ } contravariant.

$= A'_i = A_j \frac{\partial x^j}{\partial x'^i}$ } co-varient

Rank 2

$A'^{ij} = A^{kl} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l}$

$A'^i_j = A^k_l \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j}$

Complex Analysis

Cauchys Integral formula

Integration path is anticlockwise.

~~in this region~~ f(z) is analytic within and on the region C

consider any point => $z=a$ ~~not analytic~~ singular point

a is inside C

$$\boxed{f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz}$$

If a is outside

$\oint \approx 0$ since Cauchys theorem.

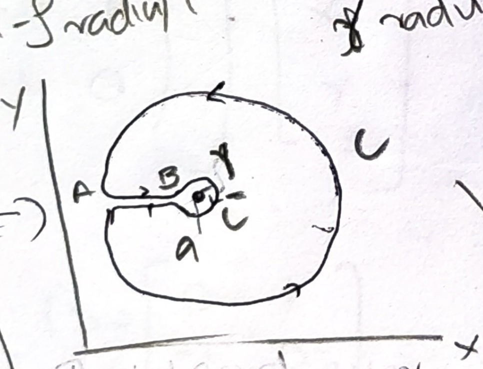
Proof:

consider the function $\frac{f(z)}{z-a}$

~~at $z=a$ not analytic at~~

analytic expect $z=a$ and δ radius

To remove singularity =>



γ radius

Now the f^n is analytic

both Now clock wise.

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C'} \frac{f(z)}{z-a}\,dz$$

(1)

$$= \int_a \frac{f(z) - f(a) + f(a)}{z-a}$$

The Integral along AB & BA cancell

$$\oint_C \frac{f(z)}{(z-a)}\,dz + \oint_{c'} \frac{f(z)}{(z-a)} = 0$$

(clock) counter clock

$\mathrm{I} + \mathrm{II} = 0$

$\mathrm{I} = -\mathrm{II}$

$$\int_a \frac{f(z)-f(a)}{z-a}\,dz + f(a)\int_a \frac{dz}{z-a}$$

$$\underbrace{\qquad\qquad}_{I_1} \qquad \underbrace{\qquad\qquad}_{I_2}$$

along C $\quad z = a + re^{i\theta}$

$z - a = re^{i\theta}$

$dz = ire^{i\theta}\,d\theta$

$$I_1 = \int_a \frac{f(z)-f(a)}{z-a}\,dz$$

$$= \int_0^{2\pi} \frac{f(a+re^{i\theta}) - f(a)}{re^{i\theta}}\, re^{i\theta}\, i\,d\theta$$

If we consider $r \to 0$ then $f(z) = f(a)$

$$= \int_0^{2\pi} \left[f(a+re^{i\theta}) - f(a) \right] i\,d\theta$$

$r \to 0$

$= 0$

To Polar limit change to $\int_0^{2\pi}$

$$f(a)\int_a \frac{dz}{z-a}$$

$$I_2 = f(a)\int_{C_1} \frac{dz}{z-a}$$

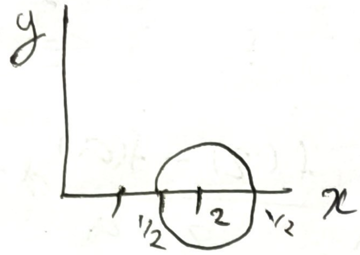
$$= f(a)\int_0^{2\pi} \frac{re^{i\theta}\,i\,d\theta}{re^{i\theta}}$$

$$= f(a)\int_0^{2\pi} i\,d\theta$$

$$= \boxed{f(a)\,2\pi i = \oint_C \frac{f(z)}{(z-a)}}$$

Proved

ex: $\int_C \frac{z}{z^2-3z+2}\,dz$ C circle $|z-2| = 1/2$



eqn of circle in complex plane.

$|x+iy-2| = 1/2$

$|(x-2)+iy| = 1/2$

eqn of circle $(x-2)^2+y^2 = (1/2)^2$

$(x-a)^2+(y-b)^2 = r^2$

eg: $|z-i| = 1/2$

$|(x-0)+i(y-1)| = 1/2$

$(0, 1)$

Step 1) find singular point

$z^2-3z+2=0$

$(z-1)(z-2)$

$z=1, 2$ Poles

$z=1$ out side the circle.

$\Rightarrow a=2$

$$\int_C \frac{z\,dz}{(z-1)(z-2)} = \int_C \frac{z(z1)}{(z-2)}\,dz$$

$$= \int_C \frac{f(z)}{(z-2)}\,dz \qquad f(z) = \frac{z}{(z-1)}$$

apply cauchys integral formula.

$$\int_c \frac{\partial f(z)}{z-a}\,dz = 2\pi i\,(f(a))$$

$$\int_c \frac{f(z)}{(z-a)}\Bigg|_{a=2} = 2\pi i \times \frac{z}{(z-1)}\Bigg|_{(z=a),(a=2)}$$

$$= 2\pi i \cdot \frac{2}{(2-1)} = \underline{\underline{4\pi i}}$$

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use cauchy's Integral formula to Calculate derivative of analytic f^n

If a f^n is analytic in a region then it's derivatives at any point $z=a$ of R is also analytic in R given by

$$\boxed{f'(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^2}dz}$$

Proof: cauchys Integral formula $\Rightarrow$

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)}dz$$

$$f'(a) = \frac{1}{2\pi i}\oint_c f(z)\frac{\partial}{\partial a}\left(\frac{1}{z-a}\right)dz$$

$$= \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^2}dz$$

$$\boxed{f^{n}(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}dz}$$

Example $I = \int_c \frac{e^{2z}}{z^4}$

$C: |z|=1 \Rightarrow I ?$

let $f(z) = e^{2z}$

$$I = \oint \frac{f(z)}{z^4}dz = \frac{2\pi i}{3!}f'''(0) = \frac{8}{3}\pi i$$

$\oint f(z)dz = 0$ cauchys ~~integral~~ theorm

If the f^n is analytic in R ~~then~~ $\Rightarrow \oint_C f(z)dz = 0$

$\oint_C f(z)dz = 0$

If f^n $\oint f(z)dz = 0 \not\Rightarrow$ analtic (not nesrorly).

eg: $\oint_C \frac{1}{z^2}\,dz$, C = unit circle centre at $x=0$

$z=0$

$\Rightarrow z = re^{i\theta}$, $r=1$

$\oint_0^{2\pi} e^{-i2\theta} \times e^{i\theta}\,d\theta$

But not not analytic at $z=0$

$= i\int_0^{2\pi} e^{-i\theta}\,d\theta = 0$ $\Rightarrow i\left[\frac{e^{i\theta}}{+i}\right]_0^{2\pi}$ $e^{i2\pi} - e^{0}$,

$-[1-1] = 0$

HW

$\oint_C z^n\,dz = ?$

If n is real Integer.
$-\infty$ to $+\infty$

(1) unit circle centre at zero

(2) unit circle centre at ~~zero~~
$z > 1$

~~Next~~ Next page:

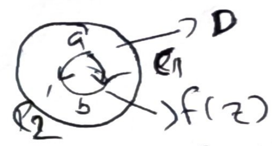
Morera theorm

converse of Cauchy's theorm

⇒ If a f^n $f(z)$ is contineus in a region R and if the Integral $f(z)d(z)$ taken around any simple closed ~~contro coun~~ contour is zero then $f(z)$ is analytic f^n inside R

Proof: $\cancel{\int_{z_0}^{z} f(z)dz = \phi(z)}$

$$\int_{a \, c_1}^{b} f(z)dz \neq \int_{a \, c_2}^{b} f(z)dz$$



integration

$\boxed{\text{analytic } f^n \text{ path is not dependend}}$

Proof:

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)}dz$$

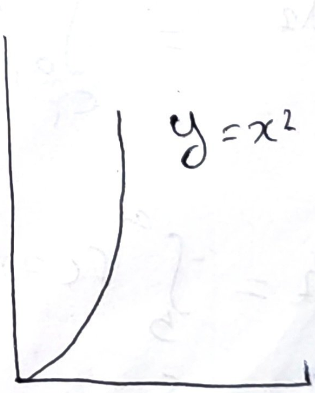
$$f(b) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-b)}dz$$

Q) $\int_c [(x+y)dx + x^2y\,dy]$ c along $y = x^2$ (0,0) to (3,9)

$= \frac{513}{2}$

$(x+x^2)dx + yy\,dy$

$$\left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{y^3}{3}\right]_{(0,0)}^{(3,9)}$$



$\frac{9}{2} + 9 + 243 = \frac{513}{2}$

along $y = x$

$$\int_C (x+y)\,dx + x^2 y\,dy = ? \qquad y = x$$

$(0,0)$ $(1,1)$

$$\int 2x\,dx + y^3\,dy$$

$$\left[\frac{2x^2}{2} + \frac{y^4}{4}\right]_{00}^{11} = \qquad 1 + \frac{1}{4} =$$

$$\int (x+x^2)\,dx + y^2\,dy$$

$(0,0)$ $(1,1)$

$$\frac{x^2}{2} + \frac{x^3}{3} + \frac{y^3}{3}\,dy$$

$y = x^2$

For analytic donot depend on path. (integration)

$\oint_C f(z)dz = 0$ (because it is analytic)

$$\Rightarrow \int_{C_1 C_2} f(z)\,dz = \int_{C_1\, a_1}^{a_2} f(z)\,dz + \int_{C_1\, a}^{b} f(z)\,dz$$

$$\int_{C_2\, a}^{b} f(z)\,dz = -\int_{C_1\, b}^{a} f(z)\,dz = \int_{C_1\, a}^{b} f(z)\,dz$$

$$\Rightarrow \int_{C_2\, a}^{b} f(z)\,dz = \int_{C_1\, a}^{b} f(z)\,dz$$

$\int_C f(z)\,dz$ you can take the straight line to calculate the integration :

---

a) $\oint_C z^n \, dz$, $C: z = re^{i\theta}$, $r=1$

$= \int_0^{2\pi} e^{in\theta} \cdot ie^{i\theta} d\theta = i\int_0^{2\pi} e^{i(n+1)\theta} d\theta$ at $n=-1$ not analytic

$= i\left[\frac{e^{i(n+1)\theta}}{(n+1)i}\right]_0^{2\pi} = \frac{1}{(n+1)i}\left[e^{i2\pi(n+1)} - e^0\right] = 0$

b) $\oint_C z^n \, dz$, $z = r_0 e^{i\theta_0} + re^{i\theta}$, $r=1$, r_0 & $\theta_0 \Rightarrow$ const.

$\oint_0^{2\pi} (r_0 e^{i\theta_0} + e^{i\theta})^n ie^{i\theta} d\theta = \cancel{\int_0^{2\pi} r_0^n e^{in\theta_0} + e^{in\theta} + 2 r_0 e^i}$

$\oint = \left[\frac{(r_0 e^{i\theta_0} + e^{i\theta})^{n+1}}{(n+1)}\right]_0^{2\pi} \neq 0$ (Wolfram alpha)

Residue

Convergancy test

let Series, $\sum_{n=0}^{\infty} f_n(z)$

absolute value

converges $\lim_{n \to \infty} \left| \frac{f(n+1)}{f(n)} \right| < 1$

diverges $\lim_{n \to \infty} \left| \frac{f(n+1)}{f_n} \right| > 1$

$= 1$ inconclusive

ratio test.

radius of convergence:

eg:
Powerseries around $z=a$

$$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n$$

Let Say $|z-a|$ = circle

calculate this

$$\lim_{n \to \infty} \left| \frac{C_{n+1} (z-a)^{n+1}}{C_n (z-a)^n} \right| = (z-a) \lim_{n \to \infty} \left| \frac{C_{n+1}}{C_n} \right|$$

$$= \frac{|z-a|}{R} < 1$$

$$\left(\frac{1}{R} = \lim_{n \to \infty} \left| \frac{C_{n+1}}{C_n} \right| \right)$$

So $|z-a| < R$ series converges
$|z-a| > R$ Diverges.

R is called Radius of convergence.

eg: $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n+3}$ find R.

$\frac{z^{n+1}}{z^n} \to \frac{1}{2} < 1$

$\frac{2}{2} < 1$

$2 < 2$

$z = 2$ radius

Soln:

$C_n = \frac{1}{2^n+3}$, $C_{n+1} = \frac{1}{2^{n+1}+3}$

$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{C_{n+1}}{C_n} \right| = \lim_{n\to\infty} \left| \frac{2^n+3}{2^{n+1}+3} \right| =$

$= \lim_{n\to\infty} \left| \frac{1+\frac{3}{2^n}}{2+\frac{3}{2^n}} \right| \Rightarrow \frac{1}{2}$ converges. <1

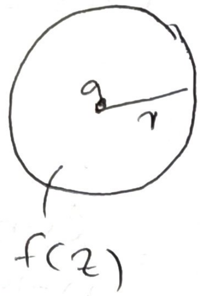
$R = 2$, radius

HW $f(z) = e^{c/2(z-\frac{1}{z})} = \sum_{n=0}^{\infty} a_n z^n$

Laurent zeries

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

Residue b_1



$$\oint_c f(z).dz = \oint_c \sum_{n0}^{\infty} a_n (z-a)^n dz + \sum_{n=1}^{\infty} \frac{b_n}{(2-a)^n} dz$$

let say m is any intiger $\pm$ varian.

$$\oint_c (z-a)^m \, dz$$

$z-a = re^{i\theta}$

$dz =$

$$= i r^{m+1} \int_0^{2\pi} e^{i(m+1)\theta} d\theta$$

residue theorm for $m = -1$ ~~$m \neq -1$~~ $m \neq -1$

$$\oint_c (z-a)^{\cancel{m}n} dz = ?$$

$= 2\pi i$

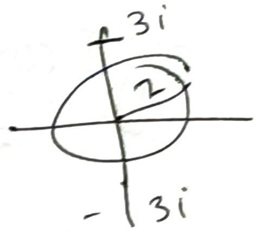
$$\oint_c (z-a)^m dz = ?$$

$= 0$

Find Residue

Example

$$I = \oint \frac{\cos z}{z^3(z^2+9)}\, dz \quad , \quad C \text{ is a circle given by } |z| = 2$$



Sol:

What are the poles of the integent

at $z=0$ & $z = \pm 3i$

So circle here includes only one singularity at $z=0$

The Laurent Serie $f(z) = \frac{\cos z}{z^3(z^2+9)}$ in the region $0<|z|<3$

So expand $f(z)$ at $z=0$

$$\frac{\cos z}{z^3(z^3+9)} = \frac{1}{9z^3}\left[\cos z \cdot \left(1+\frac{z^2}{9}\right)^{-1}\right]$$

$$= \frac{1}{9z^3}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)\left(1 - \frac{z^2}{9} + \frac{z^4}{81} \cdots\right)$$

$$\cancel{= \frac{1}{9z^3}\left(1 + \frac{z^4}{18} + \phi(z)\right)}$$

$$= \frac{1}{9z^3}\left(1 - \frac{z^2}{2} - \frac{z^2}{9} + \phi(z)\right)$$

$$= \frac{1}{9z^3}\left(1 - \frac{11z^2}{18} + \phi(z)\right)$$

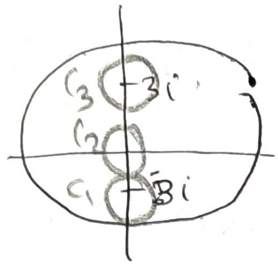
$$= \frac{1}{9z^3} - \frac{11}{162z} + \phi'(z) \Rightarrow \text{Residue } -\frac{11}{169} = b_1$$

(b_1)

$$\oint = 2\pi i \times \text{residue}$$

$$\oint \frac{\cos z}{z^3(z^2-9)} = -\frac{11\pi}{81} i$$

If $C : |z| = 4$



Now all 3 Signlar points are inside the signlar point. ~~(em~~ mutiple Connected region.

Cauchys Integral theorm.

$$\oint_C f(z)dz = \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \oint_{C_2} f(z)dz$$

For this Situation. Apply residue theorm. to the each integral on right hand side. this leads to the general. results that If $f(z)$ has number of Isolated Signlarities. with in C

then $\oint_C f(z)dz = 2\pi i$ (Sum of the Residues)

Methodes of finding Residue

(order 1)

① At Simple Pole. at $z=a$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{(z-a)}$$

$$\Rightarrow b_1 = (z-a) f(z) - \sum_{n=1}^{\infty} a_n (z-a)^{n+1}$$

~~lim z→a~~ $\lim_{z \to a} b_1 =$ II part zero,

$$\boxed{b_1 = \lim_{z \to a} (z-a) f(z)}$$

(2) Residue at pole of order n

Residue at $z = a$

$$= \frac{1}{(n-1)!} \left\{ \frac{d^{(n-1)}}{dz^{(n-1)}} (z-a)^n f(z) \right\}_{z \to a}$$

Proof Hw, Hk das

In example. I

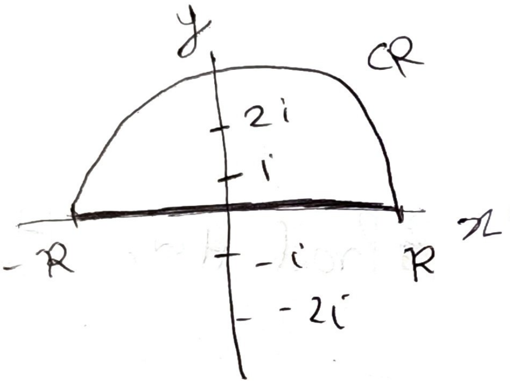
using above formula. find Reside at $z = 0$

other example — residue theorem

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$$ evalue using residue theorem

Sol: convert to: $\oint$

consider $\oint \frac{z^2\,dz}{(z^2+1)(z^2+4)} = \int f(z)\,dz$



where C contour formed by semicircle sphere of radius R and the part of Real axis from $-R$ to R

$f(z)$ has poles at $z = \pm i$ & $z = \pm 2i$ $\}$ i & $2i$ included

both of them are simple $\Rightarrow$ $(z-i) \times \frac{z^2}{(z^2+1)(z^2+2)}$

at $z = +i \Rightarrow -\frac{1}{6i}$

$\lim_{z \to i} = (z-i)\,f(z)$ $\quad \frac{z^3 - iz^2}{z^4+3z^2+2} = \frac{-i+i}{1-3+2}$

at $z = +2i \Rightarrow \frac{1}{3i}$

by residue theorem

$$\oint f(z)\,dz = 2\pi i\left(-\frac{1}{6i} + \frac{1}{3i}\right)$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{6}\right) = \pi/3$$

Now for $R \to \infty$

$$\oint_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_{CR} f(z)dz = \pi/3$$

$$I + I_2 = \pi/3$$

$= \cancel{\pi/3}$

$$\Rightarrow \lim_{R\to\infty} \underbrace{\int_{-\infty}^{\infty} f(z)dz}_{I} + \underbrace{\int_{CR} f(z)dz}_{I_2} = \frac{\pi}{3}$$

$$I_1 = \int_{-\infty}^{\infty} f(z)dz = \int_{-\infty}^{\infty} \frac{z^2 dz}{(z^2+1)(z^2+4)} = \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$$

$I_2 = \int_{CR} f(z)dz =$ for any Point on CR

$$\lim_{|z|\to\infty} \Rightarrow f(z) = 0$$

$$\Rightarrow I_2 = 0$$

$$\Rightarrow \underline{\underline{I = I_1}} \Rightarrow \oint f(z)dz = \underline{\underline{\pi/3}}$$

HW// Evaluate $= \int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$. using residue therm.

$$\int_0 \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta \qquad z = re^{i\theta} \quad r=1$$

$$e^{i\theta} \qquad z^2 = e^{2i\theta}$$

$$\int_C \frac{\frac{e^{2i\theta}+e^{-2i\theta}}{2}}{5+4\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}\,d\theta = \oint \frac{\frac{z^2+\frac{1}{z^2}}{2}}{5+4\left(\frac{z+\frac{1}{z}}{2}\right)} \qquad e^{i\theta}\,i\,d\theta = d \qquad d\theta = \frac{d}{zi}$$

$$\oint \frac{(z^4+1)\frac{1}{2}/z^2}{5+2(z^2+1)}\,\frac{dz}{iz} \Rightarrow$$

$$= \frac{1}{2i}\oint\left(\frac{(z^4+1)/z^2}{\cancel{2z^2+2z}\; 5z+\left(\frac{2z^2+2}{z}\right)}\right)dz$$

$$= \frac{1}{2i}\oint \frac{z^4+1}{5z^2+2z^3+2z}$$

Complex Analysis

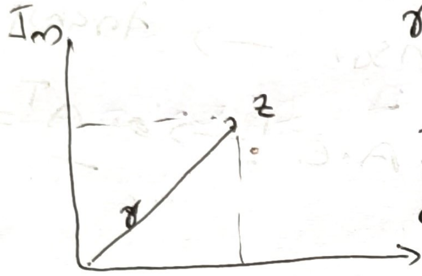
$x^2+1=0$

$x=\sqrt{-1}=i$

$\boxed{\theta = \text{in rad}}$

$Z = x+yi$

Complex plane



$r=\sqrt{Re^2+Im^2}$

$x=r\cos\theta$

$y=r\sin\theta$

$\theta=\tan^{-1}\left(\frac{y}{x}\right)$

$z\ r\cos\theta + i r\sin\theta = r(\cos\theta + i\sin\theta)$

$Z = r(e^{i\theta})$ euler funts

Complex Conjugate

$\bar{Z} = x-iy$, $\bar{z} = re^{-i\theta}$

$\overline{Z_1 \pm Z_2} = \bar{Z_1} \pm \bar{Z_2}$ ($\div$, $\times$)

$|Z_1 \times Z_2| = |Z_1| \times |Z_2|$ ($\div$)

$|Z|^2 \Rightarrow Z\bar{Z} = re^{i\theta} \cdot re^{-i\theta} = r^2$

$Z\bar{Z} = (x+iy)(x-iy) = x^2+y^2$

$\boxed{r=\sqrt{Z\bar{Z}}}$ $\Rightarrow r$ is ≥ 0 & real

Physical Applications

$Z(t) = x(t)+iy(t)$

5+8i

Velocity $= \frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}i$

$|v| = \left|\frac{dz}{dt}\right| = \sqrt{\left(\frac{dx}{dt}+i\frac{dy}{dt}\right)\left(\frac{dx}{dt}-i\frac{dy}{dt}\right)}$

$= \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

$\sqrt{25+64}$

of if $z(t)$ is given find $\frac{dz}{dt}$ then

$\left|\frac{dz}{dt}\right| = \sqrt{\frac{dz}{dt} \cdot \frac{d\bar{z}}{dt}}$

$f(z) = z^2$ elementary $f^n \Rightarrow x^i, \sqrt{x}, \sin(x), \sin^{-1}x,$

$f(y) = y^2$

euler's formula $\Leftarrow$ Tyler series of $e^{i\theta}$

$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} \Rightarrow$ mutiply absolute vale, add angle

$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

eg: $e^{2-i\pi}$

$e^2 \cdot e^{-i\pi}$

$e^2 \times (-1)$

check Problem on text book

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\cos(zi) = \dfrac{e^{z} + e^{-z}}{2} = \cosh z$

$i\sin(zi) = \dfrac{e^{z} - e^{-z}}{2i} = \sinh z$

purly imag (ixi) (-i.i)

$\cos iy = \dfrac{e^{-y} + e^{y}}{2} = \dfrac{e^{y} + e^{-y}}{2} = \cosh y$

$\sin iy = \dfrac{e^{-iy} - e^{iy}}{2i} \cdot \dfrac{xi}{xi} = i\dfrac{e^{y} - e^{-y}}{2} = i\sinh y$

$\ln(z) = \ln(x+iy)$

$= \ln(re^{i\theta}) = \ln(r) + i\theta \ln(e)$

eg: $\ln(-1)$

$-1 = 1 \times e^{i(2n\pm1)\pi}$

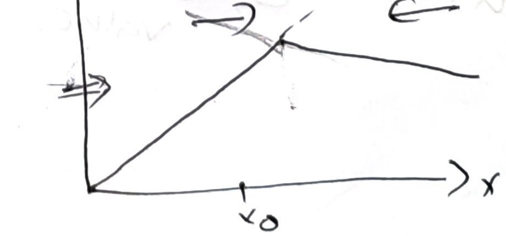
$\ln(-1) = \ln(1) + (2n\pm1)i\pi$

$\boxed{\ln z = \ln(r) + i\theta}$

$\ln a^b = b\ln a$

$\boxed{a^b = e^{b\ln a}}$

Analytic function



aways $\Leftarrow$ but have same vale for $f(x_0) \Rightarrow$ different

Diffentiabilty & continuety of function

Q1) $f(z) = z^2$

$f' = \lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

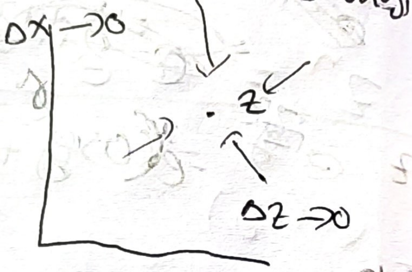
$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} = \frac{z^2 + \Delta z^2 + 2z\Delta z - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} = \Delta z + 2z \approx 2z$$

$\Delta z = \Delta x + \Delta y i$

(when)



derivative is indipendent of how we take limit $\Delta z \to 0$ => Analytic function

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Q2) $f(z) = |z|^2$ example of not analytic

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|}{\Delta z}$$

always real.

real or imaginary.

$$= \lim_{\Delta z \to 0} [Re] \frac{1}{\Delta z} \Rightarrow \Delta z = x + i\Delta y$$

case ① $\Delta x = 0$ (x fixed) $\Delta y \to 0$ $\dfrac{df}{dz} \Rightarrow$ imagina

case ② $\Delta y = 0$ (y fixed) $\Delta x \to 0$ $\dfrac{df}{dz} \Rightarrow$ real

} Non analytic(a)

Give 2 different values.

$f(z)$ unique derivative exist for the circle $z=a$ then analytic in that region

Analatic function (Holomormic or Homogenic)

A function is analytic in a region of complex plane if it has unique derivative at every point of the region.

$F(z)$ is analytic at a point $z=a$ means $f(z)$ has derivative at every point inside the small circle at

$f(z) = z^2 \Rightarrow$ Analatic (holonomic)

$f(z) = |z|^2$ = not Analytic

Check ~~whet~~ weather f^n is analytic

$$\boxed{f(z) = u(x,y) + iv(x,y)}$$

~~$z = x+iy$~~

$z = x+iy$

$$z^2 = (x+iy)^2$$
$$= x^2 + (iy)^2 + 2ixy$$
$$= \underbrace{(x^2-y^2)}_{re} + \underbrace{2ixy}_{ima_i}$$
$$z^2 = u(x,y) + v(x,y)\, i$$

eg: $z = x+iy$

$\frac{\partial u}{\partial x} = 1$ $\frac{\partial v}{\partial y} = 1$ $\Rightarrow$ Analy

$\frac{\partial u}{\partial y} = 0$ $\frac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ } these condition mean Analytic.

Cauchy Reimon Condition.

$u = u(x,y)$

$v = v(x,y)$

$f(z) = u + iv$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

Rules for partial differentiation

1) $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \quad (1)$$

$$= \frac{\partial f}{\partial z} \times 1$$

$z = x + iy$

$$\frac{\partial z}{\partial x} = 1 + 0$$

$$\frac{\partial z}{\partial y} = 0 + i$$

2) $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z} \times i \quad (2)$$

Compare ⇒

(1) ⇒ $$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

(2) $$\frac{\partial f}{\partial z} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

comparing remaining imagin

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad i\frac{\partial v}{\partial x} \quad \frac{1}{i}\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial x} = -1\frac{\partial u}{\partial y}$$

Q4) Find $\frac{df}{dz}$ assuming ~~that~~ we aproch z_0 along a strait line of slope m, and show that df/dz does not depend upon m.

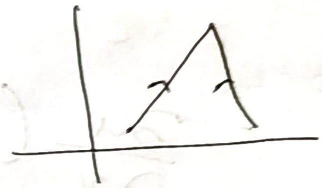
eqn of straight line

$Z_0 = x_0 + i y_0$

Straight line $(y-y_0) = m(x-x_0)$

$$m = \frac{y-y_0}{x-x_0}$$

If it is analytic do not depend on slope.



$$\frac{df}{dz} = \frac{d(u+iv)}{d(x+iy)} = \frac{du+idv}{dx+idy}$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\frac{df}{dz} = \frac{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)}{dx+idy}$$

$$= \frac{\frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} + i\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx}\right)}{1 + i\frac{dy}{dx}}$$

Cauchy reimann condition

$$\frac{df}{dz} = \frac{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}m + i\frac{\partial v}{\partial x} + im\frac{\partial v}{\partial y}}{1+im}$$

To convert all to dx

$$\frac{\partial f}{\partial z} = \frac{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} m + i \frac{\partial v}{\partial x} + i m \frac{\partial u}{\partial x}}{1 + i m}$$

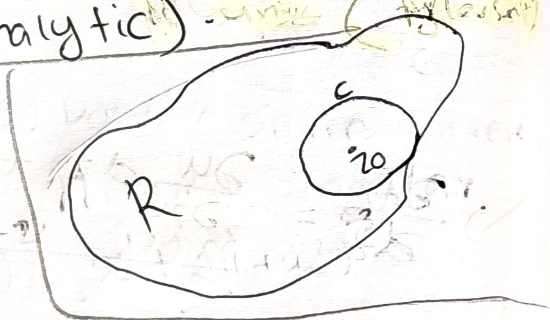
$$= \frac{\frac{\partial u}{\partial x}(1 + i m) + \frac{\partial v}{\partial x}(i - m)}{(1 + i m)} \qquad i(1+im)$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$ => Indipenden of y so any curve will have same limit => analytic

---

If $f(z)$ is analytic in a region R then all derivatives exist and can be expanded by tylor series about z_0 also power series converges inside a circle that extend to nearest singular point (point in which $f(z)$ is not analytic).

$f(z), \frac{df}{dz}, \frac{d^2f}{dz^2}, \frac{d^3f}{dz^3},$



Tyler

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$|x| < 1 \Rightarrow f(x)$ converges.

$(-1 < x < 1)$

Prove this using complex numbers.

replace x with z

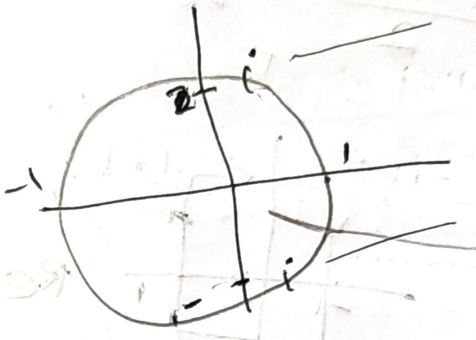
$$f(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$

let z be i (for simplicity)

$f(z) = 1 - i^2 + i^4 - i^6$ or $\frac{1}{1+i^2} = \infty$

$= 1+1+1+1+1 - - - \Rightarrow$ Diverges at $z = i$

Similary Diverses at $z = (-i)$ diverges.



Divergent point (S) sigular point

> Analytic in this region.

also aply to ~~general form~~ this form general.

If $f = \frac{1}{1+x^2} \Rightarrow$

Power series

series in the form $\sum_{n=0}^{\infty} C_n (x-a)^n$

$|x-a| < R$ converges $|x-a| = R \Rightarrow$ Radius of convergence

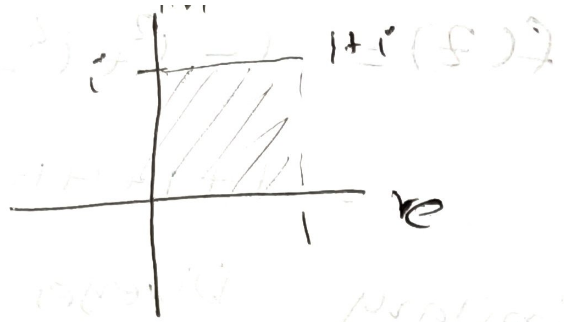
$|x-a| > R$ Diverges

Tylor series

$$F(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n \Rightarrow F(a) + f'(a)(x-a) + \frac{F''(a)(x-a)}{2!}$$

$f(z) = \overline{(z)}$

$$\int_i^{i+1} z\,dz$$



$$= \left[\frac{z^2}{2}\right]_i^{i+1} = \frac{(i+1)^2}{2} - \frac{i^2}{2} = \frac{-1+1+2i}{2} - \frac{(-1)}{2}$$

$$= 1/2 + i$$

$$\oint_c z\,dz$$

Simple curve → Not intersets C

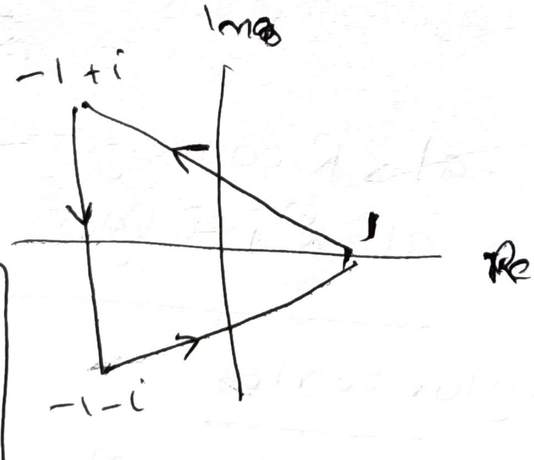
closed simple closed curve O

$$= \frac{z^2}{2}$$

$$= \frac{1}{2}\left(\left[z^2\right]_1^{1+i} + \left[z^2\right]_{1+i}^{-1+i} + \left[z^2\right]_{-1+i}^{-1} + \left[z^2\right]_{-1}^{1}\right)$$

$$= \frac{1}{2}\left[(1+i^2+2i-1) + (1+i^2+2i\cancel{)} - (1+i^2+2i) \ldots\right)$$

$$= 0$$



$$\oint_c z\,dz = \frac{z^2}{2}$$

$$\frac{1}{2}\left[z^2\Big|_1^{-1+i} + z^2\Big|_{-1+i}^{-1-i} + z^2\Big|_{-1+i}^{1}\right]$$

$$= 1+i^2-2i + 1^2+i^2+2i + \ldots = 0$$

$f(z) = z \quad , z = x + iy$

If we apply Cauchy Reimann condition

$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 1 \quad \Rightarrow$ Analytic

Let c be a simple closed curve with continuous turning points [with finite number of corners] then If f(z) is analytic inside c then $\oint_c f(z)\,d(z) = 0$

$\boxed{\oint_c f(z)\,dz = 0}$

Proof

$f(z) = u + iv$

$z = x + iy$

$\oint_c f(z)\,dz$

$= \oint_c (u+iv)(dx + i\,dy)$

$= \oint_c (u\,dx - v\,dy) + i\,(v\,dx + u\,dy)$

Green theorem

$$\oint_c P\,dx + Q\,dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$P(x,y), \; Q(x,y)$

$= \oint \left(\cancel{\frac{-\partial v}{\partial x}} - \cancel{\frac{\partial u}{\partial y}} \right) dx\,dy + i \oint_c \underbrace{\left(\cancel{\frac{\partial u}{\partial x}} - \cancel{\frac{\partial v}{\partial y}} \right)}_{\text{Cauchy reimann condition.}} dx\,dy$

$= 0$

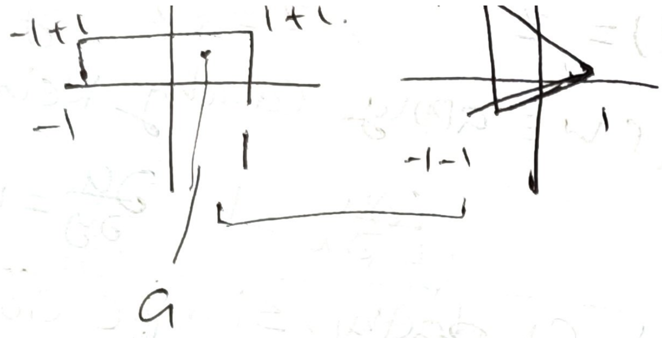
(Cauchy's theorem)

~~Simple curve~~ (

$f(z) = z^2$

$\oint_C f(z)\,dz = 0$

HW



let $f(z)$ is analytic in a closed curve 'c'

A point a $(a_0 + ia_1)$ Inside the closed Curve C

$\oint_C f(z)\,dz = 0$

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)}\,dz$$

$$\oint_C \frac{f(z)}{(z-a)}\,dz = f(a)\,2\pi i$$

* Not analytic at $z = a$

Cauchys Integral formula.

Usefull to find $f(a)$ given $f(z)$ is analytic

$z = fa$ } $\oint_C \frac{f(z)}{f}$

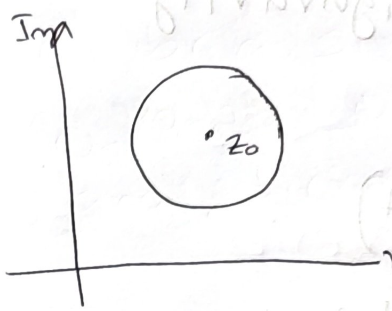
$$= \frac{1}{a(f-1)}\oint_C f(z)\,dz = 0$$

converges to zero

Lauranie Series

If $f(z)$ is analitic then tyler expania

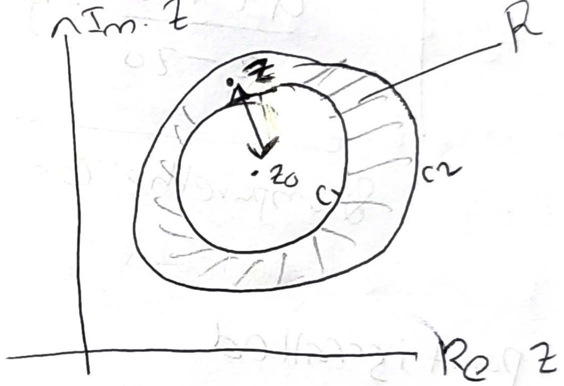
$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$



Not analitic at z_0

a term converge inside C_2

b term converge outside C_1



$\left.\begin{matrix} C_1 \\ C_2 \end{matrix}\right\}$ 2 circls. with center ~~z_0~~

$C_1 \rightsquigarrow 0$

$C_2 \rightsquigarrow \infty$

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$

$f(z)$ is analitic in

we define region R bounded by 2 concentic circle C_1, C_2

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$ tyler ①

$(z - z_0)$ Distanie

Powers of inverse of Distanie $\Leftarrow$ $+\dfrac{b_1}{z - z_0} + \dfrac{b_2}{(z - z_0)^2} + \dfrac{b_3}{(z - z_0)^3} + \cdots$

Principal part

take limit $z \to z_0$ ~~It will not tell anything for tylor.~~

$f(z)$ convergens

② term will $\Rightarrow$ f^n not defind at $z = z_0$

$z = z_0$ is Sigularity.

Tyler part of laurent series $\equiv$ analytic part = polynomia

= Rational part.

we can expand the fn in term of Laurant Series

Pole

Pole can be of any order. It is like singularity

$a_n \rightarrow$ Coe of tylor series

$b_n \rightarrow$ Inverse part (or principle part)

$$a_n = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

Proof:

$$\frac{dz\, f(z)}{z-z_0} = \oint \frac{u}{z-z_0}$$

& compare the coe.

~~examples~~ First term in principle part is called

$\frac{b_1}{\cancel{(z-z_0)}}$ is called residue. Sus

eg:

$$f(z) = \frac{\sin(z)}{z^2}$$

Expand in Laurant.

$$= \frac{1}{z^2}\left(\overbrace{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}^{\text{Tyler}}\right)$$

$$f(z) = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \cdots$$

Defin: $= a_0 + a_1(z-z_0) + \cdots + \frac{b}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$

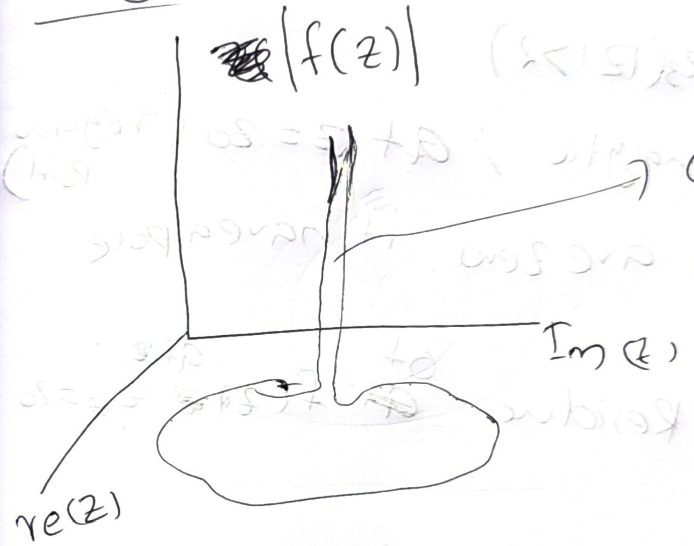
Here z_0 is zero expanding along 0

$z_0 = 0$, $b_1 = 1$

residue $\Rightarrow \frac{1}{z}$ is $z \to 0 \Rightarrow$ singularity.

we only have 1 pole which ~~are~~ is of order 1

why we call - pole



it is called pole.

there can be multiple poles

$$f(z) = \frac{1}{(z-1)^2 (z-2)^4 (z-4)}$$

at $z=1$ a pole. order 2 pole. True

at $z=2$ a pole. order 4 pole.

$(z=4)$ simple pole.

Generalization

If all the b_k are zero $\Rightarrow$ f^n $f(z)$ is analytic.

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{j=1}^{\infty} b_j (z-z_0)^{-j}$$

If b_j are zero $\Rightarrow$ analytic.

~~If $b_j \neq 0 \Rightarrow f(z)$ has a sigularity~~

~~If $b_j = 0$ for $j > n \Rightarrow$ Pole of of order N~~ } sus

---

eg: $f(z) = \dfrac{12}{z(2-z)(1+z)}$

$\left.\begin{array}{l} z=0 \\ z=2 \\ z=-1 \end{array}\right\}$ Singular point,

3 Sigular points so 3 laurent series for 3 Region.

$R_1 (0 < |z| < 1)$, $R_2 (1 < |z| < 2)$, $R_3 (|z| > 2)$

If all b's are zero $f(z)$ is analytic ; at $z = z_0$ (regular Point)

If $b_n \neq 0$ but b_n's after b_n are zero it have a pole of order n at $z = z_0$

The coe of $\dfrac{1}{z - z_0}$ is called Residue of $f(z)$ at $z_0 = z_0$

---

eg: $\dfrac{e^z}{z^3} = \dfrac{1}{z^3} + \dfrac{1}{z^2} + \dfrac{1}{2!\,z} + \dfrac{1}{3!} \ldots$

pole of order 3 at $z=0$

Residue of $\dfrac{e^z}{z^3}$ at $z=0$ is $\dfrac{1}{2!}$

$\dfrac{\sin 2z}{z^3}$ Simple Pole at $z=1$

Sin start with z So

$\dfrac{z}{z^3} \left(\dfrac{1}{z}\right) z^2$

$$H(x-a) = \begin{cases} 0 ; & x<a \\ 1 ; & x>a \end{cases} \quad x=a \text{ undefin}$$

$$\delta(x-a) = \begin{cases} 0 ; & x<a \\ 0 ; & x>a \\ \infty ; & x=a \end{cases}$$

$$\delta = \frac{dH}{dx}$$

$$\int_{-\infty}^{\infty} f(t)\, H(t)\, dt = \int_{0}^{t} f(t)\, dt$$

$$(t-a) \quad = \int_a^t f(t)\dots$$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a) = f(a)$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\Gamma(n) = (n-1)! = \int_0^{\infty} x^{(n-1)} e^{-x} dx$$

recursion relation

$$\Gamma(n+1) = n\,\Gamma(n) = \quad \Big| \quad \Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}$$

for complex

stilings apro

$$n! = n^n e^{-n} \sqrt{2\pi n}$$

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \quad \text{is symmetric} \Rightarrow B(y,x)$$

$$B(x,y) = \frac{(x-1)!\,(y-1)!}{(x+y-1)!} = \int_0^1 \tau^{y-1} (1-\tau)^{x-1} d\tau$$

$$= 2 \int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1}$$

$$= \int_0^{\infty} \frac{\tau^{x-1}}{(1+\tau)^{x+y}} d\tau$$

$\cos z = \frac{e^{iz}+e^{-iz}}{2}$; $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ $\left(e^{z} = e^{x}(\cos y + i\sin y) \right)$

$\cosh z = \frac{e^{z}+e^{-z}}{2} = \cos(iz)$; $\sinh z = \frac{e^{z}-e^{-z}}{2} = i\sin(iz)$

$\ln z = \ln r + i\theta$ | $a^{b} = e^{(b\ln a)}$

Analytic f^n

If Derivative is unique.

$f(z) = u(x,y) + iv(x,y)$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$; Cauchy-Reiman Condition

If f^n is analytic in R then Powerseries about z_0 Converges inside circle which extend to nearest Singular point.

$\oint_c f(z)\,dz = 0$, $f(z)$ is analytic inside c
Cauchy's theorm. analytic at a

$\oint_c P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$, Green's theorm

Cauchy's integral formula

$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)}\,dz$, a is inside c $\Rightarrow$ not analytic at a
if a is outside c $\Rightarrow$ analytic $\Rightarrow \oint = 0$

$z = a + re^{i\theta}$, eqn of circle.

derivative

$f^{n}(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}\,dz$

If f^n is Continuous & $\oint f(z)\,dz = 0 \Rightarrow$ Analytic

If analytic Integration is path independent

Laurant Series,

$$f(z) = \underbrace{\sum_{k=0}^{\infty} a_k (z - z_0)^k}_{\text{tylor, Polynomial, analytic}} + \underbrace{\sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}}_{\text{Principle Part}};$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Coe of $\frac{1}{z - z_0}$ = residue b_1

$$b_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

Powerseria = $f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$

$\lim_{n \to \infty} \left| \frac{f(n+1)}{f_n} \right|$ < 1 converses $\rightsquigarrow \frac{1}{R}$ - Radius of conv

> 1 diverge.

Residue theorm

$$\oint_C f(z) dz = 2\pi i \overset{\text{residue}}{b_1} \text{ or } 2\pi i (\text{Sum of Residues})$$

For simple pole $b_1 = \lim_{z \to a} f(z)(z - a)$